Pricing Asian Options For Jump Diffusions

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joint work with
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Jump Diffusions

Introduce discontinuity into the stock price dynamics

- Capture excess kurtosis, better describe catastrophic events,
- Generate implied volatility skew / smile.

\[ dS_t = \left( r - \mu \right) S_t dt + \sigma S_t dB_t + \int \mathcal{R}(y - 1) N(dt, dy) \]

\[ N(dt, dy) \text{ is a Poisson random measure with the mean measure } \lambda dt \nu(dy). \]

\[ \int \mathcal{R}(y - 1) N(dt, dy) = \sum N_t(Y - 1). \]

At the time of jump, \( S_t \rightarrow S_t - Y, \) \( Y \) has distribution \( \nu(dy) \), (jump up for \( Y > 1 \), down for \( Y < 1 \)).

\[ \mu \equiv \lambda (\xi - 1) \text{ and assume } \xi \equiv E[Y] < +\infty. \]

Example:

- Merton's model: \( \log Y \) is normal;
- Kou's model: \( \log Y \) is double exponential.
Jump Diffusions

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Assume the stock price $S$ follows jump diffusions (under the risk neutral measure $\mathbb{P}$ calibrated from the market)

$$dS_t = (r - \mu)S_t \, dt + \sigma S_t \, dB_t + S_t \int_{\mathbb{R}} (y - 1)N(dt, dy),$$

- $N(dt, dy)$ is a Poisson random measure with the mean measure $\lambda dt \, \nu(dy)$. Here $\nu(dy)$ is a finite measure on $\mathbb{R}_+$. 
  $$\int_{\mathbb{R}} (y - 1)N(dt, dy) = d \sum N_t (Y - 1).$$
- At the time of jump, $S_t \to S_t Y$, $Y$ has distribution $\nu(dy)$, (jump up for $Y > 1$, down for $Y < 1$).
- $\mu \triangleq \lambda (\xi - 1)$ and assume $\xi \triangleq \mathbb{E}[Y] < +\infty$. 

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- At the time of jump, $S_t \rightarrow S_t \, Y$, $Y$ has distribution $\nu(dy)$, (jump up for $Y > 1$, down for $Y < 1$).
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Example:

- Merton’s model: log $Y$ is normal;
- Kou’s model: log $Y$ is double exponential.
Asian Options

The value of European style continuous averaging Asian option is

$$V(S_0) \triangleq \mathbb{E}^{\mathbb{P}} \left\{ e^{-rT} \left( \zeta \cdot \left( \frac{1}{T} \int_0^T S_t dt - K_1 S_T - K_2 \right) \right)^+ \right\},$$

- $K_1$: Floating Strike, $K_2$: Fixed Strike,
- $\zeta \in \{-1, 1\}$ indicates put/call option,
- Maturity $T$ is fixed.
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$V$ satisfies a degenerate PDE with two space dimensions. Numerical solution of this PDE is difficult.
Dimension Reduction (Večer (01))

Define a process

\[ Z_t^J \triangleq \frac{X_t}{S_t}, \quad t \in [0, T], \quad Z_0^J = z = q_0 - e^{-rT} \frac{K_2}{S_0}. \]

\( X = \{ X_t, t \in [0, T] \} \) is a self-financing portfolio with dynamics

\[ dX_t = q_t dS_t + r(X_t - q_t S_t) dt, \quad X_0 = x = q_0 S_0 - e^{-rT} K_2. \]

\( q_t, t \in [0, T] \), is the number of shares invested in stock at time \( t \),

\[ q_t \triangleq \frac{1}{rT} \left( 1 - e^{-r(T-t)} \right). \]

Then

\[ X_T = \frac{1}{T} \int_0^T S_t dt - K_2. \]
Define a process

\[ Z^J_t \triangleq \frac{X_t}{S_t}, \quad t \in [0, T], \quad Z^J_0 = z = q_0 - e^{-rT} \frac{K_2}{S_0}. \]

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Then

\[ X_T = \frac{1}{T} \int_0^T S_t dt - K_2. \]

Introduce a new measure \( \mathbb{Q} \) by the Randon-Nykodym derivative

\[ \frac{d\mathbb{Q}}{d\mathbb{P}} \bigg|_{\mathcal{F}_t} = e^{-rt} \frac{S_t}{S_0}, \quad t \in [0, T], \]
Proposition (Večeř and Xu (04))

1. $V(S_0) = S_0 \cdot \mathbb{E}_z^Q[(\zeta \cdot (Z^J_t - K_1))^+]$, where

$$dZ^J_t = (q_t - Z^J_t) \left\{ \sigma dW_t + \int_{\mathbb{R}_+} \frac{y - 1}{y} [N(dt, dy) - y \nu(dy)dt] \right\}.$$
Proposition (Večeř and Xu (04))

1. \( V(S_0) = S_0 \cdot \mathbb{E}_z^Q[(\zeta \cdot (Z^J_t - K_1))^+] \), where
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\]

2. Let \( v(z, t) \) be the solution of
\[
\frac{\partial}{\partial t} v + A(t)v - \lambda \xi v + \lambda \cdot (Pv)(z, t) = 0, \quad (z, t) \in \mathbb{R} \times [0, T),
\]
\[
v(z, T) = (\zeta \cdot (z - K_1))^+,
\]
where \( A(t) := -\mu(q_t - z) \frac{\partial}{\partial z} + \frac{1}{2} \sigma^2 (q_t - z)^2 \frac{\partial^2}{\partial z^2} \) and
\[
Pv(z, t) = \int_{\mathbb{R}_+} v \left( \frac{z}{y} + q_t \frac{y-1}{y}, t \right) y\nu(dy).
\]

If \( v_t, v_z \) and \( v_{zz} \) are continuous, then \( V(S_0) = S_0 \cdot v(z, 0). \)
Proposition (Večer and Xu (04))

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\]

If \( v_t, v_z \) and \( v_{zz} \) are continuous, then \( V(S_0) = S_0 \cdot v(z, 0) \).

Our goals: Show the assumptions are satisfied, use a sequence of diffusion problems to approximate the jump diffusion problem.
Functional Operator $J$

Let us introduce the functional operator $J$ through its action on a test function $f : \mathbb{R} \times [0, T] \rightarrow \mathbb{R}_+$:

$$Jf(z, t) = \mathbb{E}_{t, z}^{Q} \left\{ e^{-\lambda \xi(T-t)} (\zeta \cdot (Z_T - K_1))^+ \right.$$ 

$$+ \int_{t}^{T} e^{-\lambda \xi(s-t)} \lambda \cdot Pf(Z_s, s) \, ds \right\},$$

in which $\mathbb{E}_{t, z}^{Q}$ is the conditional expectation and the process $Z = \{Z_t; s \geq 0\}$ is a diffusion process with the dynamics

$$dZ_s = -\mu(q_s - Z_s) \, ds + \sigma(q_s - Z_s) \, dW_s.$$ 

Recall $Pf(Z_s, s) = \int_{\mathbb{R}_+} f \left( \frac{Z_s}{y} + q_s \frac{y-1}{y}, t \right) y \nu(dy)$. 

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**Remark:** Intuitively, let $\tau_1$ be the first jump time of a Poisson process with the parameter $\lambda \xi$,

$$Jf(x, t) = \mathbb{E}_{t, z}^{Q} \left\{ (\zeta \cdot (Z_T - K_1))^+ 1_{\{\tau_1 > T-t\}} + f(S_{\tau_1}, \tau_1) 1_{\{\tau_1 \leq T-t\}} \right\}.$$
The Approximating Sequence

Using the operator $J$, let us introduce

$$v_0(z, t) \triangleq (\zeta \cdot (z - K_1^+)),$$

$$v_{n+1}(z, t) \triangleq Jv_n(z, t), \quad n \geq 0, \text{ for } (z, t) \in \mathbb{R} \times [0, T].$$
The Approximating Sequence

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We want to show:

1. $\{v_n\}_{n \geq 0}$ converges to a limit $v_\infty$,
2. $v_\infty$ is the unique classical solution, i.e. $v_\infty \in C^{2,1}$, of the following equation

$$\frac{\partial}{\partial t} v_\infty + A(t)v_\infty - \lambda \xi v_\infty + \lambda \cdot (Pv_\infty)(z, t) = 0,$$

$$v_\infty(z, T) = \left( \zeta \cdot (z - K_1) \right)^+.$$

Therefore, $V(S_0) = S_0 \cdot v_\infty(z, 0)$. 
Properties of $J$

The operator $J$ can be rewritten as:

$$Jf(z, t) = \mathbb{E}_{\xi} \left\{ e^{-\lambda \xi (T-t)} \left( \xi \cdot \left( zH_0T - t + bT - t - K_1 \right) \right) + \int_{T-t}^0 e^{-\lambda \xi s} \lambda \cdot Pf(zH_0s + bs, t + s) \, ds \right\}$$

where $H_0s = \exp((\mu - \frac{1}{2} \sigma^2) s - \sigma W_s)$ and $b_s$ is represented by $H$ and $q$.

Lemma

For any $t \in [0, T]$, $|f(z, t) - f(\tilde{z}, t)| \leq D |z - \tilde{z}|$, $z, \tilde{z} \in \mathbb{R}$.

Then $Jf$ satisfies:

$$|Jf(z, t) - Jf(\tilde{z}, t)| \leq E |z - \tilde{z}|,$$

with $E = \max\left\{ 1, D \right\}$.
Properties of $J$

$J$ maps “nice” functions to “nicer” functions.
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$$
Properties of $J$ cont.

Defining $M_f \triangleq \sup_{t \in [0, T]} f(0, t)$ and $M_{Jf} \triangleq \sup_{t \in [0, T]} Jf(0, t)$, we have $f$ and $Jf$ both satisfy linear growth conditions. Moreover,

**Lemma**

\[
M_{Jf} \leq U + \alpha \left( M_f + \frac{B}{\xi} \right),
\]

in which $\alpha = 1 - e^{-\lambda \xi T} < 1$, and $U$, $B$ are positive constants depending on $T$. 
Properties of $J$ cont.

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**Lemma**

Assume $|f(z, t) - f(\tilde{z}, t)| \leq D |z - \tilde{z}|$, for $z, \tilde{z} \in \mathbb{R}$, then

\[ |Jf(z, t) - Jf(z, s)| \leq F \left( 1 + |z| \right) (s - t)^{\frac{1}{2}}, \quad 0 \leq t < s \leq T, \]

in which $F$ is a positive constant that only depends on $\lambda$, $\xi$, $T$ and $M_f$. 
Properties of $J$ cont.

**Theorem**

 Assume function $f$ satisfies

$$|f(z, t) - f(\tilde{z}, s)| \leq D|z - \tilde{z}| + F(1 + |z|)|s - t|^{\frac{1}{2}},$$

then the function $Jf$ is the unique classical solution, i.e. $Jf \in C^{2,1}$, of

$$\mathcal{L}_{D}Jf(z, t) \triangleq \frac{\partial}{\partial t}Jf + A(t)Jf - \lambda \xi Jf = -\lambda \cdot Pf(z, t)$$

$$Jf(z, T) = (\zeta \cdot (z - K_1))^+. $$
Properties of $J$ cont.

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$$Jf(z, T) = (\zeta \cdot (z - K_1))^+.\$$

**Proof:** For any point $(z, t) \in D = [z_1, z_2] \times [0, T]$.

$$\mathcal{L}_D u(z, t) = -\lambda \cdot Pf(z, t), \quad u(z, t) = Jf(z, t), \quad (z, t) \in \partial_0 D.$$
Properties of $J$ cont.

**Theorem**
Assume function $f$ satisfies
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\]

Pervious Lemmas $\implies$ 1. $Jf$ is joint continuous, 2. $Pf(z, t)$ is Lipschitz in $z$ and Hölder continuous in $t$ uniformly in $D$. 
Properties of $J$ cont.

**Theorem**

Assume function $f$ satisfies

$$|f(z, t) - f(\tilde{z}, s)| \leq D|z - \tilde{z}| + F(1 + |z|)|s - t|^\frac{1}{2},$$

then the function $J_f$ is the unique classical solution, i.e. $J_f \in C^{2,1}$, of

$$\mathcal{L}_D J_f(z, t) \triangleq \frac{\partial}{\partial t} J_f + A(t) J_f - \lambda \xi J_f = -\lambda \cdot \mathcal{P}f(z, t)$$

$$J_f(z, T) = (\zeta \cdot (z - K_1))^+.$$

**Proof:** For any point $(z, t) \in D = [z_1, z_2] \times [0, T]$.

$$\mathcal{L}_D u(z, t) = -\lambda \cdot \mathcal{P}f(z, t), \quad u(z, t) = J_f(z, t), \quad (z, t) \in \partial_0 D.$$

Pervious Lemmas $\implies$ 1. $J_f$ is joint continuous, 2. $\mathcal{P}f(z, t)$ is Lipschitz in $z$ and Hölder continuous in $t$ uniformly in $D$. Theory of parabolic PDE $\implies$ there is an unique classical solution.
Properties of $J$ cont.

**Theorem**

Assume function $f$ satisfies

$$|f(z, t) - f(\tilde{z}, s)| \leq D|z - \tilde{z}| + F(1 + |z|)|s - t|^{\frac{1}{2}},$$

then the function $Jf$ is the unique classical solution, i.e. $Jf \in C^{2,1}$, of

$$\mathcal{L}_D Jf(z, t) \triangleq \frac{\partial}{\partial t} Jf + A(t)Jf - \lambda \xi Jf = -\lambda \cdot Pf(z, t)$$

$$Jf(z, T) = (\zeta \cdot (z - K_1))^+.$$

**Proof:** For any point $(z, t) \in D = [z_1, z_2] \times [0, T]$.

$$\mathcal{L}_D u(z, t) = -\lambda \cdot Pf(z, t), \quad u(z, t) = Jf(z, t), \quad (z, t) \in \partial_0 D.$$

Pervious Lemmas $\implies$ 1. $Jf$ is joint continuous, 2. $Pf(z, t)$ is Lipschitz in $z$ and Hölder continuous in $t$ uniformly in $D$. Theory of parabolic PDE $\implies$ there is an unique classical solution. Its representation is exactly $Jf$.
Properties of $v_n$

Lemma

1. Define $M_n = \sup_{t \in [0, T]} \{ v_n(0, t) \}$, then
   $M_n < M_\infty \triangleq \frac{U}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{B}{\xi} + K_1 < \infty$ for $n \geq 0$.
2. For $n \geq 0$, $|v_n(z, t) - v_n(\tilde{z}, t)| \leq |z - \tilde{z}|$, $z, \tilde{z} \in \mathbb{R}$.
3. $|v_n(z, t) - v_n(z, s)| \leq F_n(1 + |z|)(s - t)^{\frac{1}{2}}$, $0 \leq t < s \leq T$, in which $F_n$ are finite constants depends on $T$.
4. $\{v_n(z, t)\}_{n \geq 0}$ is a Cauchy sequence.
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3. $|v_n(z, t) - v_n(z, s)| \leq F_n(1 + |z|)(s - t)^{1/2}$, $0 \leq t < s \leq T$, in which $F_n$ are finite constants depends on $T$.

4. $\{v_n(z, t)\}_{n \geq 0}$ is a Cauchy sequence.

Remark:

- $1+2 \implies v_n(z, t) \leq M_\infty + |z| \triangleq L(z).$
Properties of \( v_n \)

Lemma

1. **Define** \( M_n = \sup_{t \in [0, T]} \{ v_n(0, t) \} \), then
   \[
   M_n < M_\infty \triangleq \frac{U}{1-\alpha} + \frac{\alpha}{1-\alpha} \frac{B}{\xi} + K_1 < \infty \text{ for } n \geq 0.
   \]

2. For \( n \geq 0 \), \(|v_n(z, t) - v_n(\tilde{z}, t)| \leq |z - \tilde{z}|, \ z, \tilde{z} \in \mathbb{R} \).

3. \(|v_n(z, t) - v_n(z, s)| \leq F_n(1 + |z|)(s - t)^{\frac{1}{2}}, \ 0 \leq t < s \leq T \), in which \( F_n \) are finite constants depends on \( T \).

4. \( \{ v_n(z, t) \}_{n \geq 0} \) is a Cauchy sequence.

**Remark:**

- \( 1+2 \implies v_n(z, t) \leq M_\infty + |z| \triangleq L(z) \).
- \( 4 \implies \) the pointwise limit for \( \{ v_n \}_{n \geq 0} \) exists, we call it \( v_\infty \). Moreover, \( v_\infty \leq L(z) \) and for any compact domain \( D \in \mathbb{R} \),

\[
|v_\infty(z, t) - v_n(z, t)| \leq M_D \left( 1 - e^{-\lambda \eta(T-t)} \right)^n,
\]

where \( M_D \) is a constant depending on \( D \) and \( \eta = \max\{ \xi, 1 \} \).
Combining 2, 3 and the Theorem for $Jf$, we have that $v_{n+1}$ is the unique classical solution, i.e. $v_{n+1} \in C^{2,1}$, of

$$\frac{\partial}{\partial t} v_{n+1} + A(t)v_{n+1} - \lambda \xi v_{n+1} + \lambda \cdot (Pv_{n})(z, t) = 0,$$

$$v_{n+1}(z, T) = (\zeta \cdot (z - K_1))^+. $$

This is a parabolic PDE with an integral term as the driving term.
Properties of $v_\infty$

Lemma

1. $v_\infty$ is a fixed point of the operator $J$.
2. $|v_\infty(z, t) - v_\infty(\tilde{z}, t)| \leq |z - \tilde{z}|$.
3. $|v_\infty(z, t) - v_\infty(z, s)| \leq F_\infty (1 + |z|)|t - s|^\frac{1}{2}$. 

Apply the Theorem on $Jf$ to $v_\infty$, we obtain

Theorem (Main Theorem)
The function $v_\infty$ is the unique classical solution, i.e. $v_\infty \in C^2_{1}$, of

$$
\frac{\partial}{\partial t} v_\infty + A(t) v_\infty - \lambda \xi v_\infty + \lambda \cdot (Pv_\infty)(z, t) = 0,
$$

$v_\infty(z, T) = (\zeta \cdot (z - K_1)) + \ldots$.

Proof: Combining 2, 3 and Theorem on $Jf$, we have that $Jv_\infty$ is the unique classical solution of the parabolic PDE with the integral term $\lambda \cdot (Pv_\infty)$. The theorem follows since $Jv_\infty = v_\infty$. 
Properties of $v_{\infty}$

Lemma

1. $v_{\infty}$ is a fixed point of the operator $J$.
2. $|v_{\infty}(z, t) - v_{\infty}(\tilde{z}, t)| \leq |z - \tilde{z}|$.
3. $|v_{\infty}(z, t) - v_{\infty}(z, s)| \leq F_{\infty} (1 + |z|) |t - s|^{\frac{1}{2}}$.

Apply the Theorem on $Jf$ to $v_{\infty}$, we obtain

Theorem (Main Theorem)

*The function $v_{\infty}$ is the unique classical solution, i.e. $v_{\infty} \in C^{2,1}$, of*

$$
\frac{\partial}{\partial t} v_{\infty} + A(t)v_{\infty} - \lambda \xi v_{\infty} + \lambda \cdot (Pv_{\infty})(z, t) = 0,
$$

$$
v_{\infty}(z, T) = (\zeta \cdot (z - K_1))^+.
$$

**Proof:** Combining 2, 3 and Theorem on $Jf$, we have that $Jv_{\infty}$ is the unique classical solution of the parabolic PDE with the integral term $\lambda \cdot (Pv_{\infty})$. The theorem follows since $Jv_{\infty} = v_{\infty}$. 
Numerical Algorithm

We solve the sequence of PDEs satisfied by $v_n$ iteratively, using the finite difference method.

- Crank-Nicolson discretization + SOR,
- trapezoidal rule to evaluate the integral $Pv_n$.

Let $\tilde{v}_n$ and $\tilde{v}_\infty$ be the numerical solutions for the discretized PDEs satisfied by $v_n$ and $v_\infty$ respectively, we have also shown

- The algorithm is stable, $\tilde{v}_n$ converges to $\tilde{v}_\infty$ uniformly and at an exponential rate.
- $\tilde{v}_\infty \to v_\infty$ as discretizations go to 0.
Numerical Algorithm

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- The algorithm is stable, $\tilde{v}_n$ converges to $\tilde{v}_\infty$ uniformly and at an exponential rate.
- $\tilde{v}_\infty \to v_\infty$ as discretizations go to 0.
Numerical Results

Table: The approximated price for continuously averaged European type Asian options for a double exponential jump model.

\( r = 0.15, \ S_0 = 100, \ T = 1, \ \rho = 0.6 \) and \( \eta_1 = \eta_2 = 25 \). Monte Carlo method uses \( 10^6 \) simulations and \( 10^3 \) time steps. "C - P" is the difference between our approximated call and put option prices. "Parity" is the difference predicted by the put-call parity. All our computations are performed on a Pentium IV 3.0 GHz machine with C++ implementation. Run times are in seconds.

<table>
<thead>
<tr>
<th>( \sigma )</th>
<th>( K_2 )</th>
<th>( \lambda )</th>
<th>Iteration Algorithm</th>
<th>Monte Carlo (Call Option)</th>
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<td>Put Option (P)</td>
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<td>3</td>
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<td>1.6</td>
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Table: The approximated price for continuously averaged European type Asian options for normal jump diffusion model.

$r = 0.15$, $S_0 = 100$, $T = 1$, $\lambda = 1$, $\mu = -0.1$ and $\sigma = 0.3$. Monte Carlo method uses $10^6$ simulations and $10^3$ time steps. "C - P" is the difference between our approximated call and put option prices. "Parity" is the difference predicted by the put-call parity. All our computations are performed on a Pentium IV 3.0 GHz machine with C++ implementation. Run times are in seconds.

<table>
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<tr>
<th>$\sigma$</th>
<th>$K_2$</th>
<th>Call Option</th>
<th>Put Option</th>
<th>C - P</th>
<th>Parity</th>
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Thanks for your attention!