Remarks on American Options for Jump Diffusions

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joint work with
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Why jump models?

\[ P^{\text{market}}(K, T) = P^{\text{BS}}(S_0, K, T, \sigma_{\text{imp}}). \]
D. S. Bates [00] studied the index option data for the 87 crash:

After the crash “out of the money (OTM) put options that provide explicit portfolio insurance against substantial downward movements in the market have been trading at high prices (as measured by implicit volatilities) . . . even more overpriced relative to OTM calls that will pay off only if the markets rises substantially.”

—— Crash Fear
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Comparing to models with stochastic volatilities, models with jumps can better explain this phenomenon.
Jump diffusions

Assume the stock price $S$ follows jump diffusions (under a risk neutral measure calibrated from the market)

$$dS_t = bS_t dt + \sigma(S_{t-}, t) S_{t-} dW_t + S_{t-} \cdot d \sum_{i=1}^{N_t} \left( e^{Y_i} - 1 \right),$$

in which

- $N_t$ is a Poisson process with rate $\lambda$ independent of $W$,
- $Y_i$ are iid. random variables with distribution $\nu(dy)$,
- $b \triangleq r - \lambda(\xi - 1)$, we assume $\xi \triangleq \mathbb{E}[e^{Y_i}] < +\infty$ so that $\{e^{-rt}S_t\}_{t \geq 0}$ is a martingale.
- We assume $\sigma(S, t) \geq \epsilon > 0$ for some positive constant $\epsilon$. 
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Examples:

- Merton’s model: $\nu(dy)$ is the dist. of normal r.v.
- Kou’s model: $\nu(dy)$ is the dist. of double exponential r.v.
The American put option

The value function of the American put option is given by

\[ V(S, t) \triangleq \max_{\tau \in I_t, T} \mathbb{E} \left[ e^{-r(\tau - t)} g(S_{\tau}) \mid S_t = S \right], \]

where \( g(S) = (K - S)^+ \).
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The domain \( \mathbb{R}^n \times [0, T) \) can be divided into the continuation region and the stopping region:

\[ C \triangleq \{(S, t) \in \mathbb{R}_+ \times [0, T) : V(S, t) > g(S)\} \]
\[ D \triangleq \{(S, t) \in \mathbb{R}_+ \times [0, T) : V(S, t) = g(S)\}. \]
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There exists an optimal exercise boundary \( s(t) \), such that

\[ C = \{S > s(t)\} \quad \text{and} \quad D = \{S \leq s(t)\}. \]
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There exists an optimal exercise boundary \( s(t) \), such that

\[ C = \{S > s(t)\} \text{ and } D = \{S \leq s(t)\}. \]

One of the optimal exercising time \( \tau_D \) is the hitting time of the stopping region \( D \).

\[ V(S_t, t) \text{ is a super-martingale,} \]
\[ V(S_{t \wedge \tau_D}, t \wedge \tau_D) \text{ is a martingale.} \]
Parabolic integro-differential equations (PIDE)

Proposition ([Pham 95], [Yang et al. 06], [Bayraktar 08])

The value function $V(S, t)$ is the unique classical solution of

$$(\partial_t + \mathcal{L}_D - (r + \lambda)) V + \lambda \int_{\mathbb{R}} V(S e^y, t) \nu(dy) = 0, \quad S > s(t),$$

$$V(S, t) = K - S, \quad S \leq s(t), \quad V(S, T) = (K - S)^+,$$

in which $\mathcal{L}_D \triangleq \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 + b S \partial_S$. Moreover,

$$\partial_S V(s(t), t) = -1, \quad and$$

$$(\partial_t + \mathcal{L}_D - (r + \lambda)) V + \lambda \int_{\mathbb{R}} V(S e^y, t) \nu(dy) \leq 0.$$
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Remark

Because of the nonlocal integral term, the classical finite difference method cannot be applied directly to solve the PIDE numerically.
Numerical algorithms

Various numerical algorithms were studied in the literature:

► [Zhang 97]
► [Andersen & Andreasen 00]
► [Kou & Wang 04]
► [Hirsa & Madan 04]
► [Cont & Voltchkova 05]
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- [d’ Halluin et al. 05]

We propose yet another numerical algorithm which

1. A slight modification of classical finite difference schemes
2. Shows the connection between jump diffusion problems and diffusion problems
The functional operator $J$

Instead of the jump diffusions, we consider the GBM:

$$dS^0_t = b S^0_t dt + \sigma S^0_t dW_t.$$
The functional operator $J$

Instead of the jump diffusions, we consider the GBM:

$$dS_t^0 = bS_t^0 dt + \sigma S_t^0 dW_t.$$ 

We introduce a functional operator $J$, for any $f : \mathbb{R}_+ \times [0, T] \to \mathbb{R}_+$,

$$J f(S, t) \triangleq \sup_{\tau \in \tilde{T}_t, T} \mathbb{E} \left[ \int_t^\tau e^{-(r+\lambda)(u-t)} \lambda \cdot P f (S_u^0, u) \, du + e^{-(r+\lambda)(\tau-t)} (K - S_\tau^0)^+ \bigg| S_t^0 = S \right],$$

in which

$$P f(S, u) \triangleq \int_{\mathbb{R}} f(e^y S, u) \nu(dy).$$
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\[
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\]

in which

\[
P f(S, u) \triangleq \int_\mathbb{R} f (e^y S, u) \nu(dy).
\]

Remark

$J f$ is the value function for an optimal stopping problem for the diffusion $S_0^0$. 
The approximating sequence

Let us iteratively define

\[ \nu_0(S, t) = (K - S)^+ , \]
\[ \nu_{n+1}(S, t) = J \nu_n(S, t), \quad n \geq 0, \quad \text{for } (S, t) \in \mathbb{R}_+ \times [0, T]. \]
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Theorem

For each \( n \geq 1 \), \( v_n \) is the unique classical solution of the following free boundary PDE:

\[ (\partial_t + \mathcal{L}_D - (r + \lambda)) v_n(S, t) = -\lambda \cdot (P v_{n-1})(S, t) , \quad S > s_n(t) , \]
\[ v_n(S, t) = K - S , \quad S \leq s_n(t) , \quad v_n(S, t) > (K - S)^+ , \quad S > s_n(t) , \]
\[ \partial_S v_n(s_n(t), t) = -1 , \quad v_n(S, T) = (K - S)^+ . \]
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\[
\begin{align*}
    \nu_0(S, t) &= (K - S)^+, \\
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For each \( n \geq 1 \), \( \nu_n \) is the unique classical solution of the following free boundary PDE:

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\begin{align*}
    \nu_n(S, t) &= \sup_{\tau \in \tilde{T}_{t, T}} \mathbb{E} \left[ e^{-r(\tau \wedge \sigma_n - t)} (K - S_{\tau \wedge \sigma_n})^+ \bigg| S_t = S \right],
\end{align*}
\]

where \( \sigma_n \) is the \( n \)-th jump time of the Poisson process \( N_t \).
The convergence of the sequence

Proposition

\( \{v_n\}_{n \geq 0} \) is an monotone increasing sequence, moreover \( v_n \leq K \).
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**Proof.**

- \( J v_0 = v_1 \geq (K - S)^+ = v_0 \),
- \( J \) maps positive functions to positive functions.

The statement follows from the induction. \( \square \)
The convergence of the sequence

Proposition
\[ \{ v_n \}_{n \geq 0} \text{ is an monotone increasing sequence, moreover } v_n \leq K. \]

Proof.

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  \item $J v_0 = v_1 \geq (K - S)^+ = v_0$,
  \item $J$ maps positive functions to positive functions.
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There exists a limit $v_\infty(S, t) \triangleq \lim_{n \to +\infty} v_n(S, t)$. 
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There exists a limit \( v_\infty(S, t) \triangleq \lim_{n \to +\infty} v_n(S, t) \).
Moreover, \( v_\infty \) is the fixed point of \( J \implies v_\infty \) is the unique classical solution of the PIDE.
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Moreover, \( v_\infty \) is the fixed point of \( J \) \( \implies \) \( v_\infty \) is the unique classical solution of the PIDE.

The verification argument \( \implies v_\infty = V \).
The convergence of the sequence

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\{v_n\}_{n \geq 0} \textit{is an monotone increasing sequence, moreover } v_n \leq K.

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There exists a limit $v_\infty(S, t) \triangleq \lim_{n \to +\infty} v_n(S, t).$

Moreover, $v_\infty$ is the fixed point of $J \implies v_\infty$ is the unique classical solution of the PIDE.

The verification argument $\implies v_\infty = V.$

Proposition

$$v_n(S, t) \leq V(S, t) \leq v_n(S, t) + K \left(1 - e^{-(r+\lambda)(T-t)}\right)^n \left(\frac{\lambda}{\lambda + r}\right)^n.$$
Numerical scheme

Defining $x = \log S$ and $u_n(x, t) = v_n(S, t)$ for $n \geq 1$, we solve the PDE satisfied by $u_n$ inductively.

- We discretize the equation by Crank-Nicolson scheme. For fixed $\Delta t$ and $\Delta x$ such that $M\Delta t = T$ and $L\Delta x = x_{max} - x_{min}$, we denote $\tilde{u}_{n}^{\ell,m}$ as the solution of the difference equation.

- $P\ u_{n-1} = \int_{\mathbb{R}} u_{n-1}(x + y, t) \rho(y) \, dy$ is a convolution integral. We evaluate it by FFT:

$$P\ u_{n-1} = (u_{n-1}^{\wedge} \cdot g^{\wedge})^{\vee}.$$ 

- We use Brennan-Schwartz algorithm (UI decomposition) or PSOR method to solve the free boundary problem.
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Complexity: Let the grid point in $x$ and $t$ be $O(N)$,

- Iterating Brennan-Schwartz: $O(N^2 \log N)$,
- Iterating PSOR: $O(N^{5/2})$. 

Convergence of the numerical solutions

Proposition

1. $\tilde{u}_n$ converges to $\tilde{u}_\infty$ uniformly, moreover

$$\max_{\ell, m} \left( \tilde{u}^\ell, m - \tilde{u}_n^\ell, m \right) \leq \left( 1 - \eta^M \right)^n \left( \frac{\lambda}{\lambda + r} \right)^n K,$$

where $\eta = \frac{1}{1 + (\lambda + r)\Delta t} \in (0, 1)$.

2. $\left| \tilde{u}^\ell, m - u(x_{\ell}, m\Delta t) \right| \to 0$, as $\Delta x, \Delta t, \Delta y \to 0$.

   The convergence is quadratic.

3. The numerical scheme is stable.
Numerical experiments

Table: American options in Kou’s model

\[ \eta_1 = \eta_2 = 25, \ p = 0.6, \ S_0 = 100. \]

The accuracy of Amin price is up to a penny. KPW 5EXP price is from [Kou et al. 05], “B-S” stands for the Brennan-Schwartz. The computations are done on the same kind of computer and run time is in seconds.

<table>
<thead>
<tr>
<th>Parameter Values</th>
<th>Amin</th>
<th>KPW 5EXP</th>
<th>Proposed Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K) (T) (\sigma) (\lambda)</td>
<td>Price</td>
<td>Value</td>
<td>Error</td>
</tr>
<tr>
<td>90 0.25 0.2 3</td>
<td>0.75</td>
<td>0.74</td>
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</tr>
<tr>
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<td>1.92</td>
<td>0</td>
</tr>
<tr>
<td>90 0.25 0.2 7</td>
<td>1.03</td>
<td>1.02</td>
<td>-0.01</td>
</tr>
<tr>
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<td>100 0.25 0.3 3</td>
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<tr>
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<td>90 1 0.2 3</td>
<td>2.91</td>
<td>2.90</td>
<td>-0.01</td>
</tr>
<tr>
<td>90 1 0.3 3</td>
<td>5.79</td>
<td>5.79</td>
<td>0</td>
</tr>
</tbody>
</table>
Numerical experiments cont.

Table: American options in Merton model

\( K=100, \ T=0.25, \ r=0.05, \ \sigma = 0.15, \ \lambda = 0.1. \) Stock price has lognormal jump distribution with \( \tilde{\mu} = -0.9 \) and \( \tilde{\sigma} = 0.45. \) dFLV comes from [D’Halluin et al. 05]. “B-S” stands for the Brennan-Schwartz.

<table>
<thead>
<tr>
<th>S(0)</th>
<th>dFLV</th>
<th>Proposed Algorithm</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>Error</td>
</tr>
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<tr>
<td>100</td>
<td>3.241</td>
<td>0.001</td>
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<tr>
<td>110</td>
<td>1.420</td>
<td>0</td>
</tr>
</tbody>
</table>
Numerical experiments cont.

**Figure:** Iteration of the price functions: $v_n(S, 0) \uparrow V(S, 0)$, $S \geq 0$. 

![Graph showing the iteration of price functions](image-url)
Numerical experiments cont.

Figure: Iteration of the Exercise Boundary: $s_n(t) \downarrow s(t)$, $t \in [0, T)$. Both $s_n(t)$ and $s(t)$ will converge to $S^* < K$ as $t \to T$. $K = 100$. 

$$s_1(t), s_2(t), s_3(t), s_i(t), i = 4, 5, 6$$

**Stock Price (S)**

$t$ (in years)

Continuation Region

Stopping Region

$S_{\text{approx}} = 98.223$
Numerical experiments cont.

Figure: Iteration of the Exercise Boundary: $s_n(t) \downarrow s(t), t \in [0, T)$. Both $s_n(t)$ and $s(t)$ will converge to $S^* < K$ as $t \to T$. $K = 100$.

Coefficients satisfies $r < \lambda \int_0^\infty (e^y - 1) \nu(dy)$. 
Regularity of the optimal exercise boundary of American Options
American options in Black-Scholes model

Let us assume that the stock price is governed by

\[ dS_t = (r - q)S_t dt + \sigma S_t dW_t, \quad q \geq 0 \text{ is the dividend.} \]
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The value of the finite horizon American put option is defined by

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$V(S, t)$ satisfies the following free boundary problem

$$\partial_t V + \frac{1}{2} \sigma^2 S^2 \partial^2_{SS} V + (r - q)S \partial_S V - rV = 0, \quad S > s(t)$$

$$V(S, t) = K - S, \quad S \leq s(t)$$

$$\partial_S V(s(t), t) = -1$$

$$V(S, T) = (K - S)^+.$$
Regularities of the value function and the boundary

It is well known that ([Karatzas & Shreve 98], [Peskir 05])

\[ V(S, t) \in C^{2,1}(\mathcal{C}), \quad V(S, t) \in C^{2,1}(\mathcal{D}), \]

\[
\lim_{s \downarrow s(t)} \frac{1}{2} \sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} \geq rK - qs(t) > 0, \quad t < T,
\]

\[ s(T-) = K, \quad r \geq q \quad s(T-) = \frac{r}{q}K, \quad r < q, \]

moreover, \( s(t) \) is continuous on \([0, T)\).
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**Question:** Is \( s(t) \) locally Lipschitz or even \( C^1 \) ?
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Moreover, \( s(t) \) is continuous on \([0, T)\).

**Question:** Is \( s(t) \) locally Lipschitz or even \( C^1 \) ?

**Answer:** Yes! Moreover, \( s(t) \) is \( C^\infty \) [Chen & Chadam 07].
American options for jump diffusions

The dynamics of stock price $S_t$ follows jump diffusions.

Proposition ([Pham 95], [Yang et al. 06], [Bayraktar 08])

The value function $V(S, t)$ is the unique classical solution of

$$(\partial_t + \mathcal{L}_D - (r + \lambda)) V + \lambda \int_{\mathbb{R}} V(S e^y, t) \nu(dy) = 0, \quad S > s(t),$$

$$V(S, t) = K - S, \quad S \leq s(t), \quad V(S, T) = (K - S)^+,$$

in which $\mathcal{L}_D \triangleq \frac{1}{2} \sigma^2 S^2 \partial_{SS}^2 + b S \partial_S$. Moreover,

$$\partial_S V(s(t), t) = -1, \quad and$$

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American options for jump diffusions
The dynamics of stock price $S_t$ follows jump diffusions.

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Probabilistic arguments?
Behavior of the free boundary

Continuity of the free boundary: [Pham 95], [Yang et al. 06], [Lamberton & Mikou 08].
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(C)

[Levendorskii 04], [Yang et al. 06], [Lamberton & Mikou 08] showed that

$$s(T^-) \triangleq \lim_{t \uparrow T} s(t) = \min\{K, S_0\} = \begin{cases} K & \text{(C) holds}, \\ S^* & \text{(C) fails} \end{cases},$$

where $S^*$ is the unique solution of the integral equation

$$l_0(S) \triangleq qS - rK + \lambda \int_{\mathbb{R}} (S e^y - K)^+\nu(dy) = 0.$$
Main results

**Our goals:**
Extend regularity properties of the free boundary \( s(t) \) to the case where (C) fails.
Our approach treats both cases simultaneously.
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Let us change back to the \( x \) variable
\[ x = \log S, \quad u(x, t) = V(S, T - t) \]
\[ b(t) = \log s(T - t). \]
We will state our main results in these variables.
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$b(t) \in H^5_8([\epsilon, T_0])$. 
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$b(t) \in H^{\frac{5}{8}}(\epsilon, T_0]$.

**Theorem**
$b(t) \in C^1(0, T]$. 
Theorem

Assume $\nu$ has a density, i.e. $\nu(dz) = \rho(z)dz$. Let $\alpha \in (0, \frac{1}{2})$. If $\rho(z)$ satisfies $\int_{-\infty}^{u} \rho(z)dz \in H^{2\alpha}(\mathbb{R}_-)$, then $b(t) \in H^{\frac{3}{2}+\alpha}([\epsilon, T])$.

If $\rho(z) \in H^{\ell-1+2\alpha}(\mathbb{R}_-)$ for $\ell \geq 1$, then $b(t) \in H^{\frac{3}{2}+\frac{\ell}{2}+\alpha}([\epsilon, T])$. 
Main results cont.

**Theorem**
Assume $\nu$ has a density, i.e. $\nu(dz) = \rho(z)dz$. Let $\alpha \in (0, \frac{1}{2})$. If $\rho(z)$ satisfies $\int_{-\infty}^{u} \rho(z)dz \in H^{2\alpha}(\mathbb{R}_-)$, then $b(t) \in H^{\frac{3}{2} + \alpha}([\epsilon, T])$.
If $\rho(z) \in H^{\ell-1+2\alpha}(\mathbb{R}_-)$ for $\ell \geq 1$, then $b(t) \in H^{\frac{3}{2}+\frac{\ell}{2}+\alpha}([\epsilon, T])$.

**Corollary**
If $\rho(z) \in C^\infty(\mathbb{R}_-)$ with $\frac{d^\ell}{dz^\ell} \rho(z)$ bounded in $\mathbb{R}_-$ for each $\ell \geq 1$, then $b(t) \in C^\infty(0, T]$. 
Theorem
Assume $\nu$ has a density, i.e. $\nu(dz) = \rho(z)dz$. Let $\alpha \in (0, \frac{1}{2})$. If $\rho(z)$ satisfies $\int_{-\infty}^{u} \rho(z)dz \in H^{2\alpha}(\mathbb{R}_-)$, then $b(t) \in H^{\frac{3}{2} + \alpha}([\epsilon, T])$. If $\rho(z) \in H^{\ell-1+2\alpha}(\mathbb{R}_-)$ for $\ell \geq 1$, then $b(t) \in H^{\frac{3}{2} + \frac{\ell}{2} + \alpha}([\epsilon, T])$.

Corollary
If $\rho(z) \in C^\infty(\mathbb{R}_-)$ with $\frac{d^\ell}{dz^\ell} \rho(z)$ bounded in $\mathbb{R}_-$ for each $\ell \geq 1$, then $b(t) \in C^\infty(0, T]$.

Remark
Free boundaries of Merton’s and Kou’s models are $C^\infty$. 
Smooth-fit property

By the smooth-fit property, for any $\epsilon > 0$, we have

$$
\frac{1}{\epsilon} \left[ \partial_{x} u(b(t), t) - \partial_{x} u(b(t - \epsilon), t - \epsilon) + e^{b(t)} - e^{b(t - \epsilon)} \right] = 0.
$$

This yields

$$
\lim_{\epsilon \to 0} \frac{b(t) - b(t - \epsilon)}{\epsilon} = -\frac{\partial_{xt}^{2} u(b(t) +, t)}{\partial_{xx}^{2} u(b(t) +, t) + e^{b(t)}}.
$$
Smooth-fit property

By the smooth-fit property, for any $\epsilon > 0$, we have

$$\frac{1}{\epsilon} \left[ \partial_x u(b(t), t) - \partial_x u(b(t - \epsilon), t - \epsilon) + e^b(t) - e^{b(t - \epsilon)} \right] = 0.$$ 

This yields

$$\lim_{\epsilon \to 0} \frac{b(t) - b(t - \epsilon)}{\epsilon} = \frac{-\partial^2_{xt} u(b(t)+, t)}{\partial^2_{xx} u(b(t)+, t) + e^b(t)}.$$ 

Two critical points to remove the condition (C): for $t \in [0, T)$$$
1. \partial^2_{xx} u(b(t)+, t) \triangleq \lim_{x \to b(t)} \partial^2_{xx} u(x, t) > -e^{b(t)},$
   (i.e. $\lim_{S \to s(t)} \partial^2_{SS} V(S, t) > 0)$
2. $\partial^2_{xt} u(b(t)+, t) \triangleq \lim_{x \to b(t)} \partial^2_{xt} u(x, t)$ exists in the classical sense and is continuous in $t$.  

Let us introduce the auxiliary function on $\mathbb{R} \times [0, T]$: 

$$
I(x, t) \triangleq qe^x - rK + \lambda \int_{\mathbb{R}} [u(x + z, t) + e^{x+z} - K] \nu(dz).
$$

- $(\partial_t - \mathcal{L} + (r + \lambda)) u(x, t) = -I(x, t)$, for $x < b(t), t \in (0, T]$,
- $x \to I(x, t)$ is strictly increasing,
- $\lim_{x \downarrow -\infty} I(x, t) = -rK$ and $\lim_{x \uparrow +\infty} I(x, t) = +\infty$. 
Let us introduce the auxiliary function on $\mathbb{R} \times [0, T]$: \( I(x, t) \triangleq qe^x - rK + \lambda \int \limits_{\mathbb{R}} \left[ u(x + z, t) + e^{x+z} - K \right] \nu(dz). \)

- \((\partial_t - \mathcal{L} + (r + \lambda)) u(x, t) = -I(x, t),\) for \(x < b(t), t \in (0, T],\)
- \(x \to I(x, t)\) is strictly increasing,
- \(\lim_{x \downarrow -\infty} I(x, t) = -rK\) and \(\lim_{x \uparrow +\infty} I(x, t) = +\infty.\)

Let us consider the level curve \(B(t) \triangleq \{ x : I(x, t) = 0, t \in [0, T] \}\)
- \(B(t) \in C^1(0, T] \cap C[0, T],\)
- \(B(t) > b(t)\) for \(t \in (0, T].\)
Corollary

\[ \partial_{xx}^2 u \left( b(t)^+, t \right) > -e^{b(t)}, \text{ for } t \in (0, T]. \]
Corollary

\[ \partial_{xx}^2 u \left( b(t) +, t \right) > -e^{b(t)}, \text{ for } t \in (0, T]. \]

Proof.

On the one hand, since \( B(t) > b(t) \) and \( x \to l(x, t) \) is strictly increasing, we have \( l(b(t), t) < 0, \text{ for } t \in (0, T]. \) On the other hand,

\[
0 = \lim_{x \downarrow b(t)} (\partial_t - \mathcal{L} + (r + \lambda)) u(x, t) \\
= -\frac{1}{2} \sigma^2 \lim_{x \downarrow b(t)} \frac{\partial^2}{\partial x^2} u(x, t) - \frac{1}{2} \sigma^2 e^{b(t)} - l(b(t), t).
\]

The statement follows from \( l(b(t), t) < 0. \) \[\Box\]
Lemma

$\partial^2_{xt} u(b(t)+, t)$ is continuous in $t \in [0, T)$. 

Numerator
Lemma
\[ \partial^2_{xt} u(b(t)+, t) \text{ is continuous in } t \in [0, T). \]

Proof.
**Step 1:** Use the Maximum Principle of the differential integral operator \( \partial_t - \mathcal{L} + (r + \lambda) \) and a test function used in Friedman & Shen 02 to show \( b(t) \in H^5_\| ([\epsilon, T_0]) \).
(Key step to drop the condition (C).)
Lemma

\[ \partial^2_{xt} u(b(t)+, t) \text{ is continuous in } t \in [0, T). \]

Proof.

Step 1: Use the Maximum Principle of the differential integral operator \( \partial_t - \mathcal{L} + (r + \lambda) \) and a test function used in Friedman & Shen 02] to show \( b(t) \in H^{5/8}([\epsilon, T_0]) \).

(Key step to drop the condition (C).)

Step 2: Since \( \frac{5}{8} > \frac{1}{2} \), a generalization of the result of [Cannon et al. 74] gives us

\[ \partial^2_{xt} u(b(t)+, t) \triangleq \lim_{x \downarrow b(t)} \partial^2_{xt} u(x, t) \]

is a continuous function with respect to \( t \).
Higher regularity

Let $\xi \triangleq x - b(t)$, $h(\xi, t) \triangleq \lambda \int_{\mathbb{R}} w(\xi + z, t) \nu(dz) + \sigma \partial_t \sigma (\partial_{xx} u - \partial_x u)$. $w(\xi, t) = \partial_t u(x, t)$ satisfies

$$
\partial_t w - \frac{1}{2} \sigma^2 \partial_{\xi\xi}^2 w - \left( \mu + b'(t) - \frac{1}{2} \sigma^2 \right) \partial_\xi w + (r + \lambda)w = h(\xi, t),
$$

$$
b'(t) = -\frac{\frac{1}{2} \sigma^2 \partial_\xi w(0, t)}{\left( \mu - r - \lambda \right) e^{b(t)} + (r + \lambda)K - \lambda \int_{\mathbb{R}} u(b(t) + z, t) \nu(dz)}.
$$

Regularity bootstrapping scheme:

$$
w \in H^{2\alpha, \alpha} \rightarrow h \in H^{2\alpha, \alpha}
$$

$$
h \in H^{2\alpha, \alpha} + b' \in H^\alpha \implies w \in H^{2\alpha + 2, \alpha + 1}
$$

$$
\implies \partial_\xi w(0, t) \in H^{\alpha + \frac{1}{2}} \implies b'(t) \in H^{\alpha + \frac{1}{2}}
$$

$\alpha$ get updated to $\alpha + \frac{1}{2}$.

($\rightarrow$ needs regularity assumption on the jump density $\rho$, because $h$ is a nonlocal integral and high order derivative of $w$ from both continuation and stopping regions may not agree on the free boundary.)
Conclusion and future research

The iterative method were also applied to

- Hedging problem [Kirch & Ruggaldier 04]
- Optimal investment problem in illiquid market [Pham & Tankov 08]
  The illiquidity is introduced when the asset prices are observed only at random times.
- Inventory control [Bayraktar & Ludkovski 08]
Non-degenerate diffusion processes dominate pure jump process.
- the value function is smooth,
- the free boundary is smooth.
Conclusion and future research cont.

Non-degenerate diffusion processes dominate pure jump process.
- the value function is smooth,
- the free boundary is smooth.

When the diffusion component degenerate or vanish
- the value function is not expected to be a classical solution,
- Smooth-fit principle may fail:
  [Boyarchenko & Levendorskii 02] and [Alili & Kyprianou 05]

Study the regularity of the value function and the optimal exercise boundary with the pure jump Lévy processes ([Caffarelli & Silvestre 08]).
Thanks for your attention!
Appendix 1: The free boundary is Hölder continuous

Lemma

For any $\epsilon > 0$, if there exists $\delta > 0$ such that for any $t_1, t_2$ satisfying $\epsilon \leq t_1 < t_2 \leq T_0$ and $t_2 - t_1 \leq \delta$

$$u(b(t_1), t) - u(b(t_1), t_1)) \leq C_\epsilon (t_2 - t_1)^\alpha, \quad t_1 \leq t \leq t_2, \quad 0 < \alpha \leq 1$$

then there exists $\delta' \in (0, \delta]$

$$b(t_1) - b(t_2) \leq C'_\epsilon (t_2 - t_1)^{\alpha/2}, \quad 0 \leq t_2 - t_1 \leq \delta'.$$
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\[ u(b(t_1), t) - u(b(t_1), t_1)) \leq C_\epsilon(t_2 - t_1)^\alpha, \quad t_1 \leq t \leq t_2, \ 0 < \alpha \leq 1 \]

then there exists $\delta' \in (0, \delta]$ such that

\[ b(t_1) - b(t_2) \leq C'_\epsilon(t_2 - t_1)^{\alpha/2}, \quad 0 \leq t_2 - t_1 \leq \delta'. \]

Use this lemma twice, one can show $b(t)$ is Hölder continuous.

1. $u(b(t_1), t) - u(b(t_1), t_1) \leq \max_s \partial_t u(b(t_1), s)(t_2 - t_1) \implies \exists \delta_1, \ b(t_1) - b(t_2) \leq C'_1(t_2 - t_1)^{\frac{1}{2}}, \ 0 \leq t_2 - t_1 \leq \delta_1.$

2. As a result, for $0 \leq t_2 - t_1 \leq \delta_1$, \n\[ \partial_t u(b(t_1), t) - \partial_t u(b(t), t) \leq C|b(t_1) - b(t)|^{\frac{1}{2}} \leq C_2(t_2 - t_1)^{\frac{1}{4}}. \]

Estimate for $\partial_t u(b(t_1), t)$ get updated, \n\[ u(b(t_1), t) - u(b(t_1), t_1) \leq C_2(t_2 - t_1)^{\frac{5}{4}}. \] Then apply the lemma again.
Appendix 1 cont. : Proof of the lemma

Consider the test function

\[
\chi(x) = \left\{ \left[ \sqrt{C_\epsilon} (t_2 - t_1) \frac{\alpha}{2} + \beta (x - b(t_1)) \right]^+ \right\}^2, \quad b(t_2) \leq x \leq b(t_1).
\]

We have \( \chi(x) > 0 \) when \( x > b(t_1) - \frac{\sqrt{C_\epsilon}}{\beta} (t_2 - t_1)^{\alpha/2} \).

We want to show \( \chi(x) \geq (u - g)(x, t), \ (x, t) \in D, \) for suitably chosen positive constant \( \beta \) and \( \delta' \in (0, \delta] \).

For any \( (x, t) \in D \), since \( (u - g)(x, t) > 0 \), we have

\[
x > b(t_1) - \frac{\sqrt{C_\epsilon}}{\beta} (t_2 - t_1)^{\alpha/2} \Rightarrow b(t_1) - b(t_2) \leq \frac{\sqrt{C_\epsilon}}{\beta} (t_2 - t_1)^{\alpha/2}.
\]
Appendix 2: $B(t) > b(t)$, $t \in (0, T]$ 

Step 1: $B(t) \geq b(t)$ If there is a $t_0 \in (0, T]$ such that $B(t_0) < b(t_0)$, since $x \rightarrow I(x, t)$ is strictly increasing, we have $I(x, t_0) > 0$ for all $x \in (B(t_0), b(t_0))$. It implies 

$$(\partial_t - \mathcal{L}_D + (r + \lambda)) u(x, t_0) = -I(x, t_0) < 0, \quad \text{for any } x \in (B(t_0), b(t_0))$$

which contradicts with $(\partial_t - \mathcal{L}_D + (r + \lambda)) u(x, t) \geq 0$.

Step 2: $B(t) > b(t)$ If there is a $t_0 \in (0, T]$ such that $B(t_0) = b(t_0)$, we have 

$$(\partial_t - \mathcal{L}_D + (r + \lambda)) (u - g)(x, t) = I(x, t) > 0, \quad \text{when } x > B(t).$$

which contradicts with the smooth-fit property at $(b(t_0), t_0)$. 

$$\partial_x (u - g)(b(t_0), t_0) > 0$$ 

contradicts with the smooth-fit property at $(b(t_0), t_0)$. 

\[ u(b(t_0), t_0) - g(t_0) = 0 \]

\[ \partial_x (u - g)(b(t_0), t_0) > 0 \]

by the Hopf Lemma.