Consumption investment optimization with Epstein-Zin utility

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What are recursive utilities?

Given a consumption stream \( c \),

\[ V_t = W(c_t, m(V_{t+1})). \]

Here \( m \) is the certainty equivalence and \( W \) is the aggregator.
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V_t = W(c_t, m(V_{t+1})).
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Here \( m \) is the certainty equivalence and \( W \) is the aggregator.

Example (Kreps-Porteus 78, Epstein-Zin 89)

\[
V_t = \left[ (1 - e^{-\delta}) c_t^{1-\frac{1}{\psi}} + e^{-\delta} \mathbb{E}_t \left[ V_{t+1}^{1-\gamma} \right]^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.
\]

Here \( \psi \): elasticity of intertemporal substitution (EIS), \( \gamma \): risk aversion

\[
EIS = \frac{\partial \ln(c_{t+1}/c_t)}{\partial r}, \quad \text{where } r = -\ln \left( \frac{u'(c_{t+1})}{u'(c_t)} \right).
\]
What are recursive utilities?

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Here $m$ is the certainty equivalence and $W$ is the aggregator.

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$$V_t = \left[ (1 - e^{-\delta}) c_t^{1 - \frac{1}{\psi}} + e^{-\delta} E_t \left[ V_{t+1}^{1-\gamma} \right]^{1 - \frac{1}{\psi}} \right]^{\frac{1}{1 - \frac{1}{\psi}}}.$$

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$$EIS = \frac{\partial \ln(c_{t+1}/c_t)}{\partial r}, \quad \text{where } r = -\ln \left( \frac{u'(c_{t+1})}{u'(c_t)} \right).$$

When $\gamma = 1/\psi$, it is the time separable von Neumann-Morgenstern utility

$$V_t^{1-\gamma} = (1 - e^{-\delta}) c_t^{1-\gamma} + e^{-\delta} E_t \left[ V_{t+1}^{1-\gamma} \right].$$
Stochastic differential utility

[Phillip Duffie Epstein 92] considered

\[ V_t = \mathbb{E}_t \left[ U(c_T) + \int_t^T f(c_u, V_u)du \right]. \]

Example (Kreps-Porteus, Epstein-Zin)

\[ f(c, \nu) = \delta \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} \left( (1-\gamma)\nu^{1-\frac{1}{\psi}} - \delta \theta \nu \right), \quad U(c) = \frac{c^{1-\gamma}}{1-\gamma}. \]

Here

\[ \theta = \frac{1-\gamma}{1-1/\psi}. \]

When \( \gamma = 1/\psi \), \( \theta = 1. \)

\[ V_t = \mathbb{E}_t \left[ e^{-\delta T} U(c_T) + \int_t^T \delta e^{-\delta s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right]. \]
Applications

- Equity premium puzzle, risk-free rate puzzle
- Excess volatility puzzle
- Credit spread puzzle
Applications

- Equity premium puzzle, risk-free rate puzzle: [Bansal-Yaron JF04]
- Excess volatility puzzle: [Benzoni-Collin Dufresne-Goldstein JFE11]
- Credit spread puzzle: [Bhamra-Kuehn-Strebulaev RFS10]

Three important ingredients in all these applications:
- Epstein-Zin utility with $\gamma > 1$ and $\psi > 1$.
- Market models where risky assets have unbounded price of risk.
- State price density (or the marginal utility of the optimal value).
Early resolution of uncertainty

When the representative agent has the Epstein-Zin utility with

$$\gamma > 1 \quad \text{and} \quad \psi > 1,$$

She prefers *early resolution of uncertainty*. 
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\[
V_t^{1-1/\psi} = U_t
\]

\[
= (1 - e^{-\delta}) \frac{c_t^{1-1/\psi}}{1 - 1/\psi} + e^{-\delta} \mathbb{E}_t[U_{t+1}]^{\frac{1}{\theta}}
\]

\[
\approx (1 - e^{-\delta}) \frac{c_t^{1-1/\psi}}{1 - 1/\psi} + e^{-\delta} \left\{ \mathbb{E}_t[U_{t+1}] + \frac{1}{2} \frac{\theta - 1}{\mathbb{E}_t[U_{t+1}]} \text{var}_t[U_{t+1}] \right\}
\]
Early resolution of uncertainty

When the representative agent has the Epstein-Zin utility with \( \gamma > 1 \) and \( \psi > 1 \),

She prefers early resolution of uncertainty.

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= (1 - e^{-\delta}) \frac{c_t^{1-1/\psi}}{1 - 1/\psi} + e^{-\delta} \mathbb{E}_t[U_{t+1}^{\theta}]^{\frac{1}{\theta}} \\
\approx (1 - e^{-\delta}) \frac{c_t^{1-1/\psi}}{1 - 1/\psi} + e^{-\delta} \left\{ \mathbb{E}_t[U_{t+1}] + \frac{1}{2} \frac{\theta - 1}{\mathbb{E}_t[U_{t+1}]} \text{var}_t[U_{t+1}] \right\}
\]

Hence asks a sizeable risk premium to compensate future uncertainty.
Utility

- [Duffie-Epstein 92]: Lipschitz aggregator
- [Duffie-Lions 92]: Markovian setting
- [Schroder-Skiadas 99]: $\theta = \frac{1-\gamma}{1-1/\psi} > 0$
- [Kraft-Seifried-Stefensen 13]: special relation between $\gamma$ and $\psi$
- [Kraft-Seiferling-Seifried working paper]: special relation is removed for model with bounded market price of risk.
Literature

**Utility**
- [Duffie-Epstein 92]: Lipschitz aggregator
- [Duffie-Lions 92]: Markovian setting
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**Market**
- [Schroder-Skiadas 99]: bounded market price of risk
- [Schroder-Skiadas 03]: factor model, \( \psi = 1 \)
- [Kraft-Seifried-Stefensen 13]: Heston model, special relation between \( \gamma \) and \( \psi \)
- [Kraft-Seiferling-Seifried working paper]: bounded price of risk
State price density

The state price density is important for applications

\[
D^* \sim \exp \left[ \int_0^t \partial_v f(c_s^*, V_s^*) \, ds \right] \partial_c f(c^*, V^*).
\]
State price density

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\[ D^* \sim \exp \left[ \int_0^\cdot \partial_v f(c_s^*, V_s^*) ds \right] \partial_c f(c^*, V^*). \]

Utility gradient [Duffie-Skiadas 94], [El Karoui-Peng-Quenez 01]:

\[ V_0 - V_0^* = \mathbb{E} \left[ \int_0^T f(c_s, V_s) - f(c_s^*, V_s^*) \, ds \right] \]
\[ \leq \mathbb{E} \left[ \int_0^T \partial_c f(c_s - c_s^*) + \partial_v f(V_s - V_s^*) \, ds \right]. \]

Define the adjoint process \( \Gamma = \exp \left( \int_0^\cdot \partial_v f(c_s^*, V_s^*) ds \right) \).

\[ V_0 - V_0^* \leq \mathbb{E} \left[ \int_0^T \Gamma_s \partial_c f(c_s - c_s^*) \right] \leq 0. \]
State price density

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\[ D^* \sim \exp \left[ \int_0^T \partial_v f(c^*_s, V^*_s) \, ds \right] \partial_c f(c^*, V^*). \]

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\[ V_0 - V_0^* \leq \mathbb{E} \left[ \int_0^T \Gamma_s \partial_c f(c_s - c^*_s) \right] \leq 0. \]

However, when \( \gamma, \psi > 1 \), \( f \) is neither Lipschitz nor jointly concave.
Our contributions

- A verification result for Epstein-Zin utility with $\gamma, \psi > 1$, extending [Hu-Imkeller-Müller 05] and [Cheridito-Hu 11].
- Methods specially designed for unbounded price of risk (risk sensitive control literature, see [Guasoni-Robertson 12]).
- Verification of the state price density.
Epstein-Zin utility via BSDE

Recall

\[ V^c_t = \mathbb{E}_t \left[ U(c_T) + \int_t^T f(c_u, V^c_u) du \right] , \quad \text{where} \]

\[ f(c, v) = \delta \frac{c^{1-\frac{1}{\psi}}}{1-\frac{1}{\psi}} ((1-\gamma)v)^{1-\frac{1}{\theta}} - \delta \theta v, \quad U(c) = \frac{c^{1-\gamma}}{1-\gamma}. \]

Consider \( Y_t = e^{-\delta t}(1-\gamma)V^c_t \) which satisfies

\[ Y_t = e^{-\delta T}c_T^{1-\gamma} + \int_t^T F(s, c_s, Y_s) ds - \int_t^T Z_s dB_s, \quad \text{where} \]

\[ F(t, c, y) = \delta \theta e^{-\delta t} c^{1-\frac{1}{\psi}} y^{1-\frac{1}{\theta}}. \]

Then \( F \) satisfies the monotonicity condition [Pardoux 99]
Epstein-Zin utility via BSDE

Recall

\[ V_t^c = \mathbb{E}_t \left[ U(c_T) + \int_t^T f(c_u, V_u^c)du \right] , \quad \text{where} \]

\[ f(c, v) = \delta \frac{c^{1-\frac{1}{\psi}}}{1 - \frac{1}{\psi}} ((1 - \gamma)v)^{1-\frac{1}{\theta}} - \delta \theta v , \quad U(c) = \frac{c^{1-\gamma}}{1 - \gamma}. \]

Consider \( Y_t = e^{-\delta \theta t} (1 - \gamma) V_t^c \) which satisfies

\[ Y_t = e^{-\delta \theta T} c_T^{1-\gamma} + \int_t^T F(s, c_s, Y_s)ds - \int_t^T Z_s dB_s , \quad \text{where} \]

\[ F(t, c, y) = \delta \theta e^{-\delta t} c^{1-\frac{1}{\psi}} y^{1-\frac{1}{\theta}}. \]

Then \( F \) satisfies the monotonicity condition \([\text{Pardoux 99}]\)

Consider \( C_a : \left\{ c : \mathbb{E} \left[ \int_0^T e^{-\delta s} c_s^{1-1/\psi} ds \right] < \infty, \mathbb{E}[c_T^{1-\gamma}] < \infty \right\} . \)

Theorem (Existence)

Suppose \( \gamma, \psi > 1. \) For any \( c \in C_a, V^c \) exists and is unique among class \( D \) processes.
Concavity

\[ c \mapsto V^c \text{ is concave, if } f(c, \nu) \text{ is jointly concave and Lipshcitz in } \nu, \]
[Duffie-Epstein 92].
Concavity

\( c \mapsto V^c \) is concave, if \( f(c, v) \) is jointly concave and Lipshcitz in \( v \), [Duffie-Epstein 92].

An orderly equivalent transformation

\[
(Y, Z) = (Y^{1/\theta}, \frac{1}{\theta} Y^{1/\theta-1} Z)/(1 - 1/\psi).
\]

\[
Y_t = e^{-\delta T} \frac{T^{1 - 1/\psi}}{1 - 1/\psi} \int_t^T \delta e^{-\delta s} \frac{c_s^{1 - 1/\psi}}{1 - 1/\psi} + \frac{1}{2} (\theta - 1) \frac{Z_s^2}{Y_s} ds - \int_t^T Z_s dB_s.
\]

This driver is jointly concave when \( \theta < 1 \).

**Theorem (Concavity)**

*When \( \gamma, \psi > 1 \), \( C_a \ni c \mapsto V^c \) is concave.*
Consumption investment problem

Financial market: $S^0$: risk free asset, $S = (S^1, \ldots, S^n)$: risky assets.

\[
\begin{align*}
\text{d}S_t^0 &= S_t^0 r(X_t) \text{d}t, \\
\text{d}S_t &= \text{diag}(S_t) \left[ (r(X_t) + \mu(X_t)) \text{d}t + \sigma(X_t) \text{d}W^\rho_t \right], \\
\text{d}X_t &= b(X_t) \text{d}t + a(X_t) \text{d}W_t, \\
\text{d}\langle W^\rho, W \rangle_t &= \rho(X_t) \text{d}t.
\end{align*}
\]

The wealth process satisfies

\[
\begin{align*}
\text{d}W_t &= W_t \left[ (r_t + \pi_t^\prime \mu_t) \text{d}t + \pi_t^\prime \sigma_t \text{d}W^\rho_t \right] - c_t \text{d}t.
\end{align*}
\]

Problem:

\[
V_0^c \to \text{Max!}
\]
The homothetic property of Epstein-Zin utility implies

\[ V_t^* = \frac{\mathcal{W}_t^{1-\gamma}}{1-\gamma} e^{Y_t}, \]

where \( Y \) satisfies the following BSDE

\[ Y_t = \int_t^T H(s, Y_s, Z_s) ds - \int_t^T Z_s d\mathcal{W}_s. \] (1)

We expect that \( V_t^* + \int_0^t f(c_s^*, V_s^*) ds \) is a martingale.

\[ H(t, y, z) = \text{quadratic in } z \]

\[ + \theta \frac{\delta \psi}{\psi} e^{-\frac{\psi}{\delta} y} \]

\[ + (1 - \gamma) r(X) + \frac{1 - \gamma}{2\gamma} \mu' \Sigma^{-1} \mu(X) \]

\[ - \delta \theta. \]
Dynamic equation

The homothetic property of Epstein-Zin utility implies

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where \( Y \) satisfies the following BSDE

\[ Y_t = \int_t^T H(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s. \tag{1} \]

We expect that \( V_t^* + \int_0^t f(c_s^*, V_s^*) ds \) is a martingale.

\[ H(t, y, z) = \text{quadratic in } z \geq 0 \]

\[ + \theta \frac{\delta \psi}{\psi} e^{-\frac{\psi}{\theta} y} \leq 0 \]

\[ + (1 - \gamma) r(X) + \frac{1 - \gamma}{2\gamma} \mu' \Sigma^{-1} \mu(X) \leq 0 \]

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Existence

Thanks to previous bounds on $H$, solution to (1) can be construction via the localization technique in [Briand-Hu 06].

**Theorem**

Suppose $\gamma, \psi > 1$ and $\mathbb{E} \left[ \int_0^T h(X_s)ds \right] > -\infty$. Then (1) admits a solution $(Y, Z)$ such that

$$
\mathbb{E}_t \left[ \int_t^T h(X_s)ds \right] - C_t \leq Y_t \leq \bar{C}_t + \log \mathbb{E}_t \left[ \exp \left( \int_t^T h(X_s)ds \right) \right].
$$

In particular, since $h \leq 0$, $Y$ is bounded from above.
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$$

In particular, since $h \leq 0$, $Y$ is bounded from above.

The candidate optimal consumption and investment strategies:

$$
\pi_t^* = \frac{1}{\gamma} \sigma_t^{-1} (\mu_t + \sigma_t \rho_t Z_t'), \quad \text{and} \quad \frac{C_t^*}{\mathcal{W}_t^*} = \delta \psi e^{-\frac{\psi}{\delta} Y_t}.
$$
Verification

\[ \text{[Hu-Imkeller-Muller] + comparison} \]

\((\pi, c)\) is \text{permissible} when \((W^\pi)^{1-\gamma} e^Y\) is of class \(D\) and \(c \in C_a\).

The same definition as in \[\text{[Hu-Cheridito 11]},\text{ when market price of risk is bounded.}\]
Verification

([Hu-Imkeller-Muller] + comparison)

\((\pi, c)\) is permissible when \((W^\pi)^{1-\gamma}e^Y\) is of class \(D\) and \(c \in C_a\).

The same definition as in [Hu-Cheridito 11], when market price of risk is bounded.

For any permissible \((\pi, c)\), let \(V = \frac{(W^\pi)^{1-\gamma}}{1-\gamma}e^Y\).

\[
V_t \geq \frac{(W^\pi_T)^{1-\gamma}}{1-\gamma} + \int_t^T f(c_s, V_s) \, ds - \int_t^T Z_s \, dB_s.
\]

Therefore \((V, Z)\) is a supersolution.

Consider the solution \((V^c, Z)\). Then comparison implies

\[
\frac{w^{1-\gamma}}{1-\gamma}e^{Y_0} = V_0 \geq V^c_0, \forall c.
\]
Verification cont

Need $V^* + \int_0^t f(c_s^*, V_s^*) ds$ is a martingale:

**Assumption**

*There exists a Lyapunov function $\phi \in C^2(E)$ such that*

i) $\phi(x) \to \infty$, as $x \to \partial E$;

ii) $\mathcal{F}[\phi]$ is bounded from below on $E$, where $\mathcal{F}$ is associated to (1).

Time separable utility: [Guasoni-Robertson 12], [Robertson-X. 14]
Verification cont

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Need to show $c^* \in C_a$.

**Assumption**

*Some moment condition of the market price of risk under a martingale measure (usually we choose the minimal martingale measure).*
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When the market price of risk is bounded, neither condition is needed.
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When the market price of risk is bounded, neither condition is needed.

**Theorem (Verification)**

Under above assumptions, $\pi^*$ and $c^*$ maximize the Epstein-Zin utility among all permissible strategies.
State price density

Recall

\[ D^* = c \exp \left[ \int_0^T \partial_v f(c_s^*, V_s^*) ds \right] \partial_c f(c^*, V^*). \]
State price density

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\[ D^* = c \exp \left[ \int_0^\cdot \partial_v f(c^*_s, V^*_s)ds \right] \partial_c f(c^*, V^*). \]

**Theorem**

The following statements hold:

i) \( \mathcal{W}^\pi D^* + \int_0^T D^*_s c_s ds \) is a supermartingale for any \((\pi, c)\);

ii) \( \mathcal{W}^* D^* + \int_0^T D^*_s c^*_s ds \) is a martingale.

Therefore

\[
\mathbb{E} \left[ \mathcal{W}^\pi_T D^*_T + \int_0^T D^*_s c_s ds \right] \leq w = \mathbb{E} \left[ \mathcal{W}^*_T D^*_T + \int_0^T D^*_s c^*_s ds \right].
\]
Example

Consider

\[ dS_t = S_t \left[ (r + \sigma \lambda X_t) dt + \sigma \sqrt{X_t} dW^\rho \right], \]
\[ dX_t = b(\ell - X_t) dt + a \sqrt{X_t} dW_t. \]

Assume \( b, \ell, r \geq 0, a, \sigma, \lambda > 0, \) and \( b\ell > \frac{1}{2} a^2. \)
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Assume \( b, \ell, r \geq 0, a, \sigma, \lambda > 0, \) and \( b\ell > \frac{1}{2}a^2. \)

The Lyapunov function can be chosen as

\[ \phi(x) = -C \log(x) + \bar{C}x, \quad \text{for sufficiently small } C, \bar{C} > 0. \]

The moment condition is satisfied when

\[ (\psi - 1) \left[ \frac{b\lambda \rho}{a} + \frac{1}{2}(\psi - (\psi - 1)\rho^2)\lambda^2 \right] < \frac{b^2}{2a^2}. \]
Example

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This condition is satisfied in the empirically relevant specifications [Liu-Pan 03]:

\[ \lambda = 0.47, \sigma = 1, b = 5, a = 0.25, \rho = -0.5, \psi = 1.5. \]
Numeric results: Optimal consumption-wealth ratio

\[ \frac{c_t^*}{W_t^*} = \delta^\psi e^{-\frac{\psi}{\sigma} Y_t}. \]
Optimal investment strategies

\[ \pi^*_t = \frac{1}{\gamma} \sigma_t^{-1} (\mu_t + \sigma_t \rho_t Z_t) \]
Log-linear transformation

Campbell-Shiller approximation (infinite horizon)

\[ e^{-\frac{\psi}{\theta}y} \approx 1 - \frac{\psi}{\theta}y, \quad \text{when } \theta \approx 0. \]
Convergence to stationary limit $\psi < 1$
Convergence to stationary limit $\psi > 1$
Localization technique of Briand and Hu

Consider a quadratic BSDE with unbounded terminal condition:

$$Y_t = \xi + \int_t^T f(t, Y_t, Z_t) dt - \int_t^T dW_t.$$ 

Suppose that we have an a priori bounds on $Y$:

$$\underline{Y} \leq Y \leq \overline{Y}, \quad \text{for locally bounded } \underline{Y}, \overline{Y}.$$
Localization technique of Briand and Hu

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\[ \underline{Y} \leq Y \leq \bar{Y}, \quad \text{for locally bounded } \underline{Y}, \bar{Y}. \]

1. With truncated \( \xi^n \), BSDE admits solution \((Y_n, Z_n)\) with \( Y_{n+1} \geq Y_n \) [Kobylanski 00].
2. For a reducing sequence \((\sigma_m)\), construct

\[ (Y^m)^{\sigma_m} := \lim_{n \to \infty} (Y_n)^{\sigma_m}. \]

3. Since \((Y^k)^{\sigma_m} = (Y^{k-1})^{\sigma_m}\) for \( k > m \), define

\[ Y^{\sigma_m} := (Y^m)^{\sigma_m}. \]

4. Send \( m \to \infty \) and verify the terminal condition.
Conclusion

This paper covers three aspects, which are important for applications:

- $\gamma, \psi > 1$
- unbounded market value of risk
- state price density
- When $\psi > 1$, finite horizon problem converges slowly to its infinite horizon limit

Future studies:

- Duality
  Variational formulation of the recursive utility [Geoffard 95], [El Karoui-Peng-Quenez 97]
- Equilibrium
  [Dumas-Uppal-Wang 00], [Bank-Riedel 01]
Thanks for your attention!