Optimal consumption and investment for stochastic differential utility
a duality approach

Hao Xing

London School of Economics

joint work with

Anis Matoussi

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What are recursive utilities?

Given a consumption stream $c$,

$$U_t = W(c_t, m(U_{t+1})).$$

Here $m$ is the certainty equivalence and $W$ is the aggregator.
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Example (Kreps-Porteus 78, Epstein-Zin 89)

$$CE_t = \left[ (1 - e^{-\delta})c_t^{\frac{1}{\psi}} + e^{-\delta}\mathbb{E}_t \left[ CE_{t+1}^{1-\gamma} \right]^{\frac{1}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}}.$$

Here $\psi$: elasticity of intertemporal substitution (EIS), $\gamma$: risk aversion
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\]

Here \( \psi \): elasticity of intertemporal substitution (EIS), \( \gamma \): risk aversion

When \( \gamma = 1/\psi \), it is the time separable von Neumann-Morgenstern utility

\[
CE_t^{1-\gamma} = (1 - e^{-\delta}) c_t^{1-\gamma} + e^{-\delta} \mathbb{E}_t[CE_{t+1}^{1-\gamma}],
\]

\[
CE_0^{1-\gamma} \sim \mathbb{E} \left[ \sum_{t=1}^{N} e^{-\delta t} c_t^{1-\gamma} \right].
\]
Stochastic differential utility (continuous time)
[Duffie Epstein 92]

\[ U_t = \mathbb{E}_t \left[ U(c_T) + \int_t^T f(c_s, U_s)ds \right]. \]
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Example (Kreps-Porteus, Epstein-Zin)

\[ f(c, U) = \delta \frac{c^{1-\frac{1}{\psi}}}{1 - \frac{1}{\psi}} ((1 - \gamma)U)^{1-\frac{1}{\theta}} - \delta \theta U, \quad U(c) = \frac{c^{1-\gamma}}{1 - \gamma}. \]

Here

\[ \theta = \frac{1 - \gamma}{1 - 1/\psi}. \]

When \(\gamma = 1/\psi\), \(\theta = 1\).

\[ U_t = \mathbb{E}_t \left[ e^{-\delta T} U(c_T) + \int_t^T \delta e^{-\delta s} \frac{c_s^{1-\gamma}}{1 - \gamma} ds \right]. \]
Applications

- Equity premium puzzle, risk-free rate puzzle
- Excess volatility puzzle
- Credit spread puzzle

Three important ingredients in all these applications:

- Epstein-Zin utility with $\gamma > 1$ and $\psi > 1$.
- Market models where risky assets have unbounded price of risk.
- State price density (or marginal utility of indirect utility).
Applications

▶ Equity premium puzzle, risk-free rate puzzle: [Bansal-Yaron JF04]
▶ Excess volatility puzzle: [Benzoni-Collin Dufresne-Goldstein JFE11]
▶ Credit spread puzzle: [Bhamra-Kuehn-Strebulaev RFS10]

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▶ Market models where risky assets have unbounded price of risk.
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Early resolution of uncertainty

When the representative agent has Epstein-Zin utility with

\[ \gamma > 1 \quad \text{and} \quad \psi > 1, \]

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\[ U_t^{1-1/\psi} = V_t \]
\[ = (1 - e^{-\delta}) \frac{c_t^{1-1/\psi}}{1 - 1/\psi} + e^{-\delta} \mathbb{E}_t [V_{t+1}]^{\frac{1}{\psi}} \]
\[ \approx (1 - e^{-\delta}) \frac{c_t^{1-1/\psi}}{1 - 1/\psi} + e^{-\delta} \left\{ \mathbb{E}_t [V_{t+1}] + \frac{1}{2} \frac{\theta - 1}{\mathbb{E}_t[V_{t+1}]} \text{var}_t[V_{t+1}] \right\} \]
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\]

Hence asks a sizeable risk premium to compensate future uncertainty.
Difficulty 1

\[ U_t^c = \mathbb{E}_t \left[ U(c_T) + \int_t^T f(c_u, U_u^c)du \right], \quad \text{where} \]

\[ f(c, U) = \delta \frac{c^{1-\frac{1}{\psi}}}{1 - \frac{1}{\psi}} ((1 - \gamma)U)^{1-\frac{1}{\theta}} - \delta \theta U, \quad U(c) = \frac{c^{1-\gamma}}{1-\gamma}. \]

When \( \theta < 0 \), \( f \) has super-linear growth in \( U \).
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\( c \) is admissible if \( \mathbb{E} \left[ \int_0^T c_t^\ell dt + c_T^\ell \right] < \infty \) for all \( \ell \in \mathbb{R} \).
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[Schroder-Skiadas 99], [Kraft-Seifried-Stefensen 13], [Kraft-Seiferling-Seifried 15]
HJB equation is nonlinear

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**Optimal control of BSDE**

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Optimal control of BSDE

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  unbounded coefficients, but restricted admissible set

These are all primal approaches.
\[ D^* \sim \exp \left[ \int_0^\cdot \partial_v f(c_s^*, U_s^*) ds \right] \partial_c f(c^*, U^*). \]
\[ D^* \sim \exp \left[ \int_0^T \partial_v f(c_s^*, U_s^*) \, ds \right] \partial_c f(c^*, U^*). \]

Utility gradient [Duffie-Skiadas 94], [El Karoui-Peng-Quenez 01]:

\[ U_0 - U_0^* = \mathbb{E} \left[ \int_0^T f(c_s, U_s) - f(c_s^*, U_s^*) \, ds \right] \]

\[ \leq \mathbb{E} \left[ \int_0^T \partial_c f(c_s - c_s^*) + \partial_v f(U_s - U_s^*) \, ds \right]. \]

Define the adjoint process \( \Gamma = \exp \left( \int_0^T \partial_v f(c_s^*, U_s^*) \, ds \right) \).

\[ U_0 - U_0^* \leq \mathbb{E} \left[ \int_0^T \Gamma_s \partial_c f(c_s - c_s^*) \right] \leq 0. \]
Difficulty 3

\[ D^* \sim \exp \left[ \int_0^\cdot \partial_v f(c^*_s, U^*_s) ds \right] \partial_c f(c^*, U^*). \]

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However, when \( \gamma, \psi > 1, \) \( f \) is not jointly concave.
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However, when \( \gamma, \psi > 1 \), \( f \) is not jointly concave.

- \( \mathcal{W}^{\pi} D^* + \int_0^\cdot D_s^* c_s \, ds \) is a supermartingale for any \((\pi, c)\);
- \( \mathcal{W}^{\pi^*} D^* + \int_0^\cdot D_s^* c_s^* \, ds \) is a martingale for optimal \((\pi^*, c^*)\).
Duality for time separable utility

\[ U_0 = \sup_{c \in C} \mathbb{E} \left[ \int_0^T U(c_t) \, dt \right] \text{ subject to } \mathbb{E} \left[ \int_0^T D_t c_t \, dt \right] \leq w \]

\[ \leq \sup_{c \in C} \mathbb{E} \left[ \int_0^T U(c_t) - yD_t c_t \, dt \right] + wy \]

\[ \leq \inf_{D \in D} \mathbb{E} \left[ \int_0^T V(yD_t) \, dt \right] + wy, \]

where \( V(D) = \sup_c \{ U(c) - Dc \} \).
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Mathematically beautiful and requires minimal assumptions.
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Mathematically beautiful and requires minimal assumptions.

For stochastic differential utility:

- Q1: What is the dual problem?
- Q2: Is gradient of indirect utility minimizer of the dual problem?
Variational representation

Assume $f$ is concave in $c$ and convex in $U$.
(For Epstein-Zin, equivalent to $\gamma \psi > 1$)

$$U^c_t = \mathbb{E}_t \left[ \int_t^T f(c_s, U^c_s) ds \right].$$

Let $F(c, \nu)$ be the concave dual of $U \mapsto f(c, U)$,

$$f(c, U) = \sup_{\nu} \{ F(c, \nu) - \nu U \}.$$
Variational representation

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$U^c$ has the following variational representation ([Geoffard 95], [El Karoui-Peng-Quenez 97])

$$U^c_t = \text{ess sup}_\nu \mathbb{E}_t \left[ \int_t^T e^{-\int_t^s \nu u du} F(c_s, \nu_s) ds \right].$$
Duality bound

\[
\sup_c U_0^c = \sup_c \sup_\nu \mathbb{E} \left[ \int_0^T e^{-\int_0^s \nu_u du} F(c_s, \nu_s) ds \right]
\]
Duality bound

\[ \sup_c U^c_0 = \sup_c \sup_{\nu} \mathbb{E} \left[ \int_0^T e^{-\int_0^s \nu(u) du} F(c_s, \nu_s) ds \right] \]

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Duality bound

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When \( G \) is power in \( D \), define \( g(D, \nu) = \sup_{\nu} \{ G(D, \nu) - \nu \mathbb{E} \} \).
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\[
\leq \inf_D \left\{ \mathbb{E}[V_0^{y,D}] + \nu y \right\}
\]

\[
V_t^{y,D} = \mathbb{E}_t \left[ \int_t^T g(D_s, kV_s^{y,D}) ds \right]
\]
Double Fenchel-Legendre transformation

Recursive

Primal \( U_T(c), f(c, u) \)

Variational

Dual \( V_T(d), g(d, \nu) \)

concave conjugate in \( u \)

convex conjugate in \( c \)

convex conjugate in \( \nu \)
Epstein-Zin utility

Primal: stochastic differential utility

\[ U_t^c = \mathbb{E}_t \left[ \frac{c_T^{1-\gamma}}{1-\gamma} + \int_t^T f(c_s, U_s^c) ds \right] \]

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\[ C = \{ c \text{ can be financed by nonnegative } \mathcal{W}^\pi, \, U^c \text{ exists} \}. \]

Dual: stochastic differential dual

\[ V_t^{yD} = \mathbb{E}_t \left[ \gamma \frac{y_{DT}}{1 - \gamma} \left( \frac{\gamma - 1}{\gamma} \right) + \int_t^T g(y_{D_s}, V_s^{yD}) \, ds \right] \]

\[ g(D, V) = \frac{\delta \psi}{\psi - 1} D^{1 - \psi} \left( \frac{1 - \gamma}{\gamma} V \right)^{1 - \frac{\gamma \psi}{\theta}} - \frac{\delta \theta}{\gamma} V. \]

\[ D = \{ DW^\pi + \int_0^t D_s c_s \, ds \text{ is a supartingale}, \, V^{yD} \text{ exists} \}. \]
Sufficient condition for existence and uniqueness

Proposition

Let $\gamma, \psi > 1$.

1. When $\mathbb{E}\left[ \int_0^T c_s^{1-1/\psi} ds + c_T^{1-\gamma} \right] < \infty$,
   then there exists a unique $U^c$ of class $(D)$.

2. When $\mathbb{E}\left[ \int_0^T D_s^{1-\psi} ds + D_T^{\gamma-1} \right] < \infty$,
   then there exists a unique $V^{yD}$ of class $(D)$, for any $y > 0$. 
Sufficient condition for existence and uniqueness

**Proposition**

Let $\gamma, \psi > 1$.  

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   then there exists a unique $V^{yD}$ of class (D), for any $y > 0$.

**Remark:** Instead of considering equation for $U^c$, consider

$Y_t = e^{-\delta \theta t}(1 - \gamma) U^c_t$.

$$Y_t = e^{-\delta \theta T} c_T^{1-\gamma} + \int_t^T F(s, c_s, Y_s) ds - \int_t^T Z_s dB_s,$$

where $F(t, c, y) = \delta \theta e^{-\delta t} c^{1-\frac{1}{\psi}} y^{1-\frac{1}{\psi}}$.

$F$ satisfies the **monotonicity condition** [Pardoux 99]
Consumption investment problem

Financial market: $S^0$: risk free asset, $S = (S^1, \ldots, S^n)$: risky assets.

$$dS^0_t = S^0_t r(X_t) \, dt,$$
$$dS_t = \text{diag}(S_t) \left[(r(X_t) + \mu(X_t)) \, dt + \sigma(X_t) \, dW^\rho_t\right],$$
$$dX_t = b(X_t) \, dt + a(X_t) \, dW_t,$$
$$d\langle W^\rho, W \rangle_t = \rho(X_t) \, dt.$$

The wealth process satisfies

$$d\mathcal{W}_t = \mathcal{W}_t \left[(r_t + \pi_t'\mu_t) \, dt + \pi_t'\sigma_t \, dW^\rho_t\right] - c_t \, dt.$$

Problem:

$$U^c_0 \rightarrow \text{Max!}$$
Dynamic equation

The homothetic property of Epstein-Zin utility implies

$$U^*_t = \frac{\mathcal{W}_t^{1-\gamma}}{1-\gamma} e^{Y_t} \quad \text{and} \quad V^*_t = \frac{\gamma}{1-\gamma} (yD_t)^{\frac{\gamma-1}{\gamma}} e^{Y_t/\gamma}$$

where $Y$ satisfies the following BSDE

$$Y_t = \int_t^T H(s, Y_s, Z_s) ds - \int_t^T Z_s dW_s. \quad (1)$$

$U^* + \int_0^t f(c^*_s, U^*_s) ds$ and $V^* + \int_0^t g(yD^*_s, V^*_s) ds$ are martingales.

$$H(t, y, z) = \text{quadratic in } z$$

$$+ \theta \frac{\delta \psi}{\psi} e^{-\frac{\psi}{\theta} y}$$

$$+ (1 - \gamma) r(X) + \frac{1 - \gamma}{2\gamma} \mu' \Sigma^{-1} \mu(X)$$

$$- \delta \theta.$$
Dynamic equation

The homothetic property of Epstein-Zin utility implies

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\( U^* + \int_0^t f(c_s^*, U_s^*) ds \) and \( V^* + \int_0^t g(yD_s^*, V_s^*) ds \) are martingales.

\[ H(t, y, z) = \text{quadratic in } z \geq 0 \]
\[ + \theta \frac{\delta \psi}{\psi} e^{-\frac{\psi}{\delta}} y \leq 0 \]
\[ + (1 - \gamma) r(X) + \frac{1 - \gamma}{2\gamma} \mu' \Sigma^{-1} \mu(X) =: h(X) \leq h_{max} \]
\[ - \delta \theta. \]
Existence

Thanks to previous bounds on $H$, solution to (1) can be construction via the localization technique in [Briand-Hu 06].

Proposition

Suppose $\gamma, \psi > 1$ and $\mathbb{E} \left[ \int_0^T h(X_s) ds \right] > -\infty$. Then (1) admits a solution $(Y, Z)$ such that

$$
\mathbb{E}_t \left[ \int_t^T h(X_s) ds \right] - C_t \leq Y_t \leq C_t + \log \mathbb{E}_t \left[ \exp \left( \int_t^T h(X_s) ds \right) \right].
$$

In particular, since $h \leq h_{\text{max}}$, $Y$ is bounded from above.
Existence

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$$\mathbb{E}_t \left[ \int_t^T h(X_s) ds \right] - C_t \leq Y_t \leq \overline{C}_t + \log \mathbb{E}_t \left[ \exp \left( \int_t^T h(X_s) ds \right) \right].$$

In particular, since $h \leq h_{\text{max}}$, $Y$ is bounded from above.

The candidate optimal strategies:

$$\pi^*_t = \frac{1}{\gamma} \sigma^{-1}_t (\mu_t + \sigma_t \rho_t Z'_t) \quad \frac{c^*_t}{W^*_t} = \delta^\psi e^{-\frac{\psi}{\theta}} Y_t,$$

$$dD_t^*/D_t^*$$ is also given in $Z$. 

Verification

Only need to show primal and dual candidate lead to the same value (i.e. no duality gap)
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(i.e. no duality gap)
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Assumption

There exists a Lyapunov function \( \phi \in C^2(E) \) such that

i) \( \phi(x) \to \infty, \text{ as } x \to \partial E; \)

ii) \( \mathcal{F}[\phi] \text{ is bounded from below on } E, \text{ where } \mathcal{F} \text{ is associated to } (1). \)
Verification

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Lyapunov function exists
\[ \implies \text{non-explosion [Stroock-Varadhan 97 Chap. 10]} \]
\[ \implies \text{some exponential local martingale is martingale.} \]
Main result

Theorem
Let $\gamma, \psi > 1$ and the previous assumption hold. Then

1. $c^*$ and $\pi^*$ are optimal consumption and investment strategy.

2. The marginal indirect utility $D^*$ is the minimizer of the dual problem.

3. $W^{\pi^*} D^* + \int_0^\cdot D^*_s c^*_s ds$ is a martingale.
Main result

Theorem

Let $\gamma, \psi > 1$ and the previous assumption hold. Then

1. $c^*$ and $\pi^*$ are optimal consumption and investment strategy.

2. The marginal indirect utility $D^*$ is the minimizer of the dual problem.

3. $\mathcal{W}^{\pi^*} D^* + \int_0^\cdot D^*_s c^*_s ds$ is a martingale.

Remark:

$D^*$ can be interpreted as the least favourable completion

[Karatzas-Lehoczky-Shreve-Xu 91]
Example: Kim-Omberg model

\[
dS_t/S_t = (r(X_t) + \mu(X_t))dt + \sigma dB_t,
\]
\[
dX_t = -bX_t dt + aW_t,
\]
where \( r(X) = r_0 + r_1 X \) and \( \mu(X) = \sigma(\lambda_0 + \lambda_1 X) \).

Main theorem holds under either of the following parameter restrictions:

1. \( r_1 = 0 \) and \( -b + \frac{1-\gamma}{\gamma} a\lambda'_1 \sigma' \Sigma^{-1} \sigma \rho < 0; \)
   OR
2. \( \lambda'_1 \sigma' \Sigma^{-1} \sigma \lambda_1 > 0. \)
Example: Kim-Omberg model

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Main theorem holds under either of the following parameter restrictions:

1. \( r_1 = 0 \) and \(-b + \frac{1-\gamma}{\gamma} a\lambda_1'\sigma'\Sigma^{-1}\sigma\rho < 0;\)

   OR

2. \( \lambda_1'\sigma'\Sigma^{-1}\sigma\lambda_1 > 0.\)

Lyapunov function is

\[ \phi(x) = C|x|^2. \]
Heston model

\[
dS_t / S_t = (r(X_t) + \lambda X_t)dt + \sqrt{X_t}dB_t,
\]
\[
dX_t = b(\ell - X_t)dt + a\sqrt{X_t}dW_t,
\]

where \( b \geq 0, a > 0, b\ell > \frac{1}{2}a^2, r(X) = r_0 + r_1X. \)

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Main theorem holds under either of the following parameter restrictions:

1. \( r_1 > 0; \)
   
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Lyapunov function is

\[ \phi(x) = -C\log(x) + \bar{C}x. \]
Conclusion

- Introduce a dual problem for SDU

- Solve consumption-investment problem under minimal conditions.

  \[ \gamma, \psi > 1, \text{ unbounded market price of risk} \]

- State price density (marginal indirect utility) is the minimizer of the dual problem.
Thanks for your attention!