Existence and convergence of Glosten-Milgrom equilibria

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joint work with
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Market with insiders

What is a market?
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For example, securities exchanges.

Market maker is the middle man who hold inventories that help to match buyers and sellers arriving at different times.
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Classical asset price theory has little to say about how orders are matched.

Problem is even more difficult when agents have different information.

For example, *insiders* who has private information

- How completely do prices reflect insider information?
- How large are insider profits?
- How does the market maker behave?
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**Market microstructure** theory: [O’Hara]
Kyle model

Kyle (85) studied a market with single risky asset and obtained equilibrium between

- strategic informed investor [insider] whose trades move prices;
- other (uninformed) investors [noise trader] have random demand;
- a risk neutral market maker observe the aggregated demand and set the price as the conditional expectation of the risky asset.
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Kyle’s model is quite influential:

- Kyle’s $\lambda^*$ measures market depth.
- Continuous time model has been studied by Back(92).
- Connection to filtration enlargement: Jacod, Jeulin, Yor, Protter . . .
- Mathematical Finance: Pikovsky & Karatzas, Imkeller, Schweizer, Ankirchner, Amendinger, Monoyios, Campi, Cetin, Danilova . . .
Glosten-Milgrom model

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Glosten-Milgrom (85) proposed a model

- insider, noise trader, and market maker.
- market maker treats each order individually. Bid/ask prices
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- insider, noise trader, and market maker.
- market maker treats each order individually. Bid/ask prices

Bid-ask spread exists because market maker wants to recoup the losses suffered in trading with informed trader.
Kyle meets Glosten and Milgrom


- Noise trades in Glosten-Milgrom is modeled by difference of two independent Poisson with intensity $\beta$ and jump size $\delta$.
- Noise trades in Kyle-Back is modeled by a Brownian motion.
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Back and Baruch showed

1. When $\delta \downarrow 0$ and $\beta \uparrow \infty$,

   Glosten-Milgrom equilibria $\implies$ Kyle-Back equilibrium.

2. When $\delta$ is small,

   bid-ask spread $\sim 2\delta \lambda^*$. 
Our contributions

In Back-Baruch 04:

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- hard-to-check conditions for convergence of equilibria.
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- hard-to-check conditions for convergence of equilibria.

Our contributions:

1. Optimal strategies for insider is explicitly constructed in Glosten-Milgrom.
2. A point process bridge is constructed.
3. Use weak convergence to remove technical assumptions for convergence.
Model

Interest rate 0.

One risky asset whose fundamental value is \( \tilde{v} \).

\( \tilde{v} \) has two states: high and low: 0 and 1 resp.

The fundamental value will be revealed at time 1.

Three types of market participants:

- **Noisy/liquidity traders**: total demand \( Z = Z^B - Z^S \),
  \( Z^B/\delta \) and \( Z^S/\delta \) are independent Poisson with intensity \( \beta \).

- **Informed trader/insider**: observes \( \tilde{v} \) at time 0,
  net order \( X = X^B - X^S \).

- **Market maker**: observe the aggregated order process \( Y = X + Z \)
  and set the price at \( \mathbb{E}[\tilde{v} \mid \mathcal{F}_t^Y] \).
Admissibility

A function $p : \delta \mathbb{Z} \times [0, 1] \rightarrow [0, 1]$ is a pricing rule if

$$y \mapsto p(y, t)$$

is strictly increasing.
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Insider’s filtration $\mathcal{F}_I$ contains filtration generated by $Z$ and $\tilde{v} + \mathcal{G}$.

The insider’s strategy $(X^B, X^S)$ is admissible if

i) $X^B$ and $X^S$, with $X^B_0 = X^S_0 = 0$, are $\mathcal{F}_I$–adapted increasing point processes with jump size $\delta$;

ii) $Z^B$ and $X^B$ (resp. $Z^S$ and $X^S$) never jump at the same time;

iii) $X^B$ and $X^S$ admit $(\mathcal{F}_I, \mathbb{P})$–intensities $\theta^B$ and $\theta^S$. 
Admissibility

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Insider can either contributes or cancels noise trader’s order.

$X^B = X^{B,B} + X^{B,S}$, where

- $X^{B,B}$: buy orders which compensate $Z^B$,
- $X^{B,S}$: buy orders which cancel some orders of $Z^S$. 
Noise trader
Market maker
A Glosten-Milgrom equilibrium is \((p, X^B, X^S)\) such that

i) Given \((X^B, X^S)\), \(p(Y_t, t) = E[\tilde{v} | \mathcal{F}_t^Y]\) for \(t \in [0, 1]\);

ii) given \(p\), \((X^B, X^S)\) maximizes the expected profit.
Insider’s problem

Insider maximizes profit associated to $(X^B, X^S)$

$$\int_0^1 X_t \, dp(Y_t, t) + (\tilde{v} - p(Y_1, 1))X_1.$$
Insider’s problem

Insider maximizes profit associated to \((X^B, X^S)\)

\[
\int_0^1 X_t^- dp(Y_t, t) + (\tilde{v} - p(Y_1, 1))X_1.
\]

This can be rewrite as

\[
\int_0^1 (\tilde{v} - p(Y_t, t))dX_t^B - \int_0^1 (\tilde{v} - p(Y_t, t))dX_t^S
\]

\[
= \int_0^1 (\tilde{v} - p(Y_t- + \delta, t))dX_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_t-, t))dX_t^{B,S}
\]

\[
- \int_0^1 (\tilde{v} - p(Y_t- - \delta, t))dX_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_t-, t))dX_t^{S,B}.
\]

Hence ask/bid prices can be defined

\[
a(Y_t- \delta, t) := p(Y_t- \delta, t)
\]

\[
b(Y_t- \delta, t) := p(Y_t- \delta, t)
\]
Insider’s problem

Insider maximizes profit associated to \((X^B, X^S)\)

\[\int_0^1 X_t \, dp(Y_t, t) + (\tilde{v} - p(Y_1, 1))X_1.\]

This can be rewritten as

\[\int_0^1 (\tilde{v} - p(Y_t, t))dX_t^B - \int_0^1 (\tilde{v} - p(Y_t, t))dX_t^S\]

\[= \int_0^1 (\tilde{v} - p(Y_{t-} + \delta, t))d\theta_t^{B,B} + \int_0^1 (\tilde{v} - p(Y_{t-}, t))d\theta_t^{B,S}
- \int_0^1 (\tilde{v} - p(Y_{t-} - \delta, t))d\theta_t^{S,S} - \int_0^1 (\tilde{v} - p(Y_{t-}, t))d\theta_t^{S,B}
+ \text{martingales.}\]

Hence ask/bid prices can be defined

\[a(Y_{t-}, t) := p(Y_{t-} + \delta, t) \quad \text{and} \quad b(Y_{t-}, t) := p(Y_{t-} - \delta, t).\]

Since \(y \mapsto p\) is increasing, \(a > b\).
Value function

Given \( \tilde{\nu} \), the value function for the insider is

\[
V(\tilde{\nu}, y, t) := \sup_{\theta^{i,j}, i,j \in \{B, S\}} \mathbb{E}_P \left[ \int_t^1 (\tilde{\nu} - a(Y_{u-}, u)) d\theta_u^{B,B} + \int_t^1 (\tilde{\nu} - p(Y_{u-}, u)) d\theta_u^{B,S} \\
- \int_t^1 (\tilde{\nu} - b(Y_{u-}, u)) d\theta_u^{S,S} - \int_t^1 (\tilde{\nu} - p(Y_{u-}, u)) d\theta_u^{S,B} \right] \bigg| \tilde{\nu}, Y_t = y.
\]
HJB equation

The value function $V(\tilde{v}, y, t)$ is expected to satisfy,

$$V_t + (V(y + \delta, t) - 2V(y, t) + V(y - \delta, t)) \beta$$

$$+ \sup_{\theta^B, B \geq 0} [V(y + \delta, t) - V(y, t) + (\tilde{v} - a(y, t)) \delta] \theta^B, B$$

$$+ \sup_{\theta^B, S \geq 0} [V(y, t) - V(y - \delta, t) + (\tilde{v} - p(y, t)) \delta] \theta^B, S$$

$$+ \text{sell side} = 0.$$ 

The system reduces to

$$V_t + (V(y + \delta, t) - 2V(y, t) + V(y - \delta, t)) \beta = 0,$$

$$V(y + \delta, t) - V(y, t) + (\tilde{v} - p(y + \delta, t)) \delta \leq 0,$$  (1)

$$V(y - \delta, t) - V(y, t) - (\tilde{v} - p(y - \delta, t)) \delta \leq 0.$$  (2)
Several observations

\[ V_t + (V(y + \delta, t) - 2V(y, t) + V(y - \delta, t)) \beta = 0, \]
\[ V(y + \delta, t) - V(y, t) + (\tilde{v} - p(y + \delta, t))\delta \leq 0, \]  \hspace{1cm} (3)
\[ V(y - \delta, t) - V(y, t) - (\tilde{v} - p(y - \delta, t))\delta \leq 0. \]  \hspace{1cm} (4)

- \( \theta^B \cdot > 0 \) only when (3) = 0; \( \theta^S \cdot > 0 \) only when (4) = 0;
- When one eqn. is identity, the other eqn. is strict inequality;
- \( p \sim \partial_y V \) where \( V \) is a harmonic function. Therefore

\[ p(y, t) = \mathbb{E}_\mathbb{P} [ p(Z_1, 1)| Z_t = y] . \]
Characterization of equilibrium

For $z \in \delta \mathbb{Z}$, let $P^z(y) = p(y, 1) = \mathbb{I}_{y \geq z}$ and

$$p^z(y, t) := \mathbb{P}[Z_1 \geq z \mid Z_t = y].$$

We expect $[\tilde{\nu} = 1] = [Y_1 \geq y_\delta]$ for some $y_\delta$.

Theorem

$(p^{y_\delta}, X^B, X^S)$ is a Glosten-Milgrom equilibrium if

i) $[Y_1 \geq y_\delta] = [\tilde{\nu} = 1]$ $\mathbb{P}$-a.s. for some $y_\delta \in \delta \mathbb{Z}$;

ii) $X^S \equiv 0$ on $[\tilde{\nu} = 1]$ ($X^B \equiv 0$ on $[\tilde{\nu} = 0]$).

iii) $Y = Z + X^B - X^S$ where $Y^B/\delta$ and $Y^S/\delta$ are independent, $\mathcal{F}^Y$–adapted Poisson processes with intensity $\beta$;
Characterization of equilibrium

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**Theorem**

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iii) \( Y = Z + X^B - X^S \) where \( Y^B / \delta \) and \( Y^S / \delta \) are independent, \( \mathcal{F}^Y \)-adapted Poisson processes with intensity \( \beta \);

**Proof:** Verification.

\[
\mathbb{E}_\mathbb{P}[\tilde{v} \mid \mathcal{F}_t^Y] = \mathbb{P}[Y_1 \geq y_\delta \mid \mathcal{F}_t^Y] = \mathbb{P}[Z_1 \geq y_\delta \mid Z_t = Y_t] = p^{y_\delta}(Y_t, t).
\]
Point process bridge

Assume $\delta = 1$.

We want to construct on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$

$$Y = Z^B - Z^S + X^B I_{l} - X^S I_{l^c},$$

$l \in \mathcal{F}_0$ and two point processes $(X^B, X^S)$, such that

i) $l = [Y_1 \geq y_1] \mathbb{P}$–a.s.;

ii) $Y^B$ and $Y^S$ are independent poisson with intensity $\beta$. 

$$G_t = Z^B_t \vee \sigma_Y$$
Point process bridge

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We want to construct on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in [0,1]}, \mathbb{P})$

$$Y = Z^B - Z^S + X^B\mathbb{1}_I - X^S\mathbb{1}_{I^c},$$

$I \in \mathcal{F}_0$ and two point processes $(X^B, X^S)$, such that

i) $I = [Y_1 \geq y_1] \mathbb{P}$-a.s.;

ii) $Y^B$ and $Y^S$ are independent poisson with intensity $\beta$.

Similar to initial filtration enlargement where $\mathcal{G}_t = \mathcal{F}_t^Z \vee \sigma([Y_1 \geq y_1])$. 
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Similar to initial filtration enlargement where $\mathcal{G}_t = \mathcal{F}_t^Z \vee \sigma([Y_1 \geq y_1])$.

However, standard theory gives

$$Y = Z^B - Z^S + \text{absolute cont. part.}$$

The absolute cont. part gives $\mathcal{G}$-intensities of $Y^B$ and $Y^S$. 
Explicit construction

Let us focus on $I$ and before the first jump of $Y$.
Recall $p(y, t) = \mathbb{P}[Z_1 \geq y_1 | Z_t = y]$.

We want to construct, before the first jump of $Y$:

$$Y^B = Z^B + X^{B, B} \text{ with intensity } \beta \frac{p(1, t)}{p(0, t)} > \beta;$$

$$Y^S = Z^S - X^{B, S} \text{ with intensity } \beta \frac{p(-1, t)}{p(0, t)} < \beta.$$
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$X^{B,B}$ and $X^{B,S}$ are constructed to match the desired intensities.

> For $X^{B,B}$, construct $\nu_1$ with $\mathbb{P}(\nu_1 > t) = \exp \left( \beta \int_0^t \left[ 1 - \frac{p(1, u)}{p(0, u)} \right] du \right)$;

> for $X^{B,S}$, accept jumps of $Z^S$ at the rate $\beta \frac{p(-1, t)}{p(0, t)}$. 
Explicit construction

Let us focus on \( I \) and before the first jump of \( Y \).

Recall \( p(y, t) = \mathbb{P}[Z_1 \geq y_1 \mid Z_t = y] \).

We want to construct, before the first jump of \( Y \):

\[
Y^B = Z^B + X^{B,B} \quad \text{with intensity} \quad \beta \frac{p(1, t)}{p(0, t)} > \beta;
\]

\[
Y^S = Z^S - X^{B,S} \quad \text{with intensity} \quad \beta \frac{p(-1, t)}{p(0, t)} < \beta.
\]

\( X^{B,B} \) and \( X^{B,S} \) are constructed to match the desired intensities.

- For \( X^{B,B} \), construct \( \nu_1 \) with \( \mathbb{P}(\nu_1 > t) = \exp \left( \beta \int_0^t \left[ 1 - \frac{p(1, u)}{p(0, u)} \right] du \right) \);
- for \( X^{B,S} \), accept jumps of \( Z^S \) at the rate \( \beta \frac{p(-1, t)}{p(0, t)} \).

Both can be achieved by introducing a sequence of iid uniform \([0, 1]\) r.v.
First jump
Existence of Glosten-Milgrom equilibrium

Proposition

When $\mathbb{P}(I) = p(0,0)$, then $Y$ constructed above satisfies

i) $[Y_1 \geq y_1] = I \mathbb{P}$-a.s.;

ii) $Y^B$ and $Y^S$ are independent Poisson with intensity $\beta$ under the natural filtration of $Y$. 

Existence of Glosten-Milgrom equilibrium

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Theorem (Existence)
If there exists $y_\delta \in \delta \mathbb{Z}$ such that

$$\mathbb{P}(Z_1 \geq y_\delta) = \mathbb{P}(\tilde{v} = 1),$$

and $\mathcal{F}_t$ includes filtration generated by $\tilde{v}$, $Z$, and a sequence of iid uniform. Then there exists a Glosten-Milgrom equilibrium.
Kyle-Back model

In Kyle-Back model, demand of noise trader $Z$ is a Brownian motion.

When $\tilde{v} = 0$ or 1, set $y_0 = \Phi^{-1}(1 - P(\tilde{v} = 1))$ and pricing function

$$p^0(y, t) := P^0_y[W_{1-t} \geq y_0].$$

Then the equilibrium demand satisfies

$$Y = W + \mathbb{I}_{[\tilde{v}=1]} \int_0^\cdot \partial_y \log p^0(Y_s, s) \, ds + \mathbb{I}_{[\tilde{v}=0]} \int_0^\cdot \partial_y \log(1 - p^0(Y_s, s)) \, ds, \, ds.$$

$Y$ is Brownian motion conditional on $\mathbb{I}_{[W_1 \geq y_0]}$.

The insider’s strategy is the additional drift in the enlarged filtration.
Convergence

When \( \beta^\delta = (2\delta^2)^{-1} \),

\[ Z^{B,\delta} - Z^{S,\delta} \xrightarrow{\mathcal{L}} W. \]
Convergence

When $\beta^\delta = (2\delta^2)^{-1}$,

$$Z^{B,\delta} - Z^{S,\delta} \overset{\mathcal{L}}{\to} W.$$ 

Theorem (Convergence)

For any $\tilde{v}$ satisfying $\mathbb{P}(\tilde{v} = 1) \in (0, 1)$, $\exists (\tilde{v}^\delta)_{\delta > 0} \overset{\mathcal{L}}{\to} \tilde{v}$, s.t.

G-M equilibrium $(p^\delta, X^{B,\delta}, X^{S,\delta})$ exists whose fundamental value of risky asset is $\tilde{v}^\delta$.

When $\beta^\delta = (2\delta^2)^{-1}$, as $\delta \to 0$, G-M equilibria $\to$ K-B equilibrium:


$$\lim_{\delta \downarrow 0} \frac{1}{\delta} \left( a^\delta(y, t) - p^\delta(y, t) \right) = \lim_{\delta \downarrow 0} \frac{1}{\delta} \left( p^\delta(y, t) - b^\delta(y, t) \right) = \partial_y p^0(y, t).$$

ii) When $\tilde{v} = 1$, $X^{B,\delta} \overset{\mathcal{L}}{\to} B^0$; when $\tilde{v} = 0$ $X^{S,\delta} \overset{\mathcal{L}}{\to} S^0$. 
Proof

Let us consider \( \tilde{v} = 1 \).

\[
\mathbb{P}^{\delta,H} = \text{Law}(Y^\delta | Y^\delta_1 \geq y^\delta).
\]

Then \( \mathbb{P}^{\delta,H} \xrightarrow{\mathcal{L}} \mathbb{P}^{0,H} \) if

a) finite dimensional distributions converge;

b) \( (\mathbb{P}^{\delta,H})_\delta \) is tight, which is equivalent to

- uniform bounded,
- equi-continuous.
Conclusion

- Construct a point process bridge;
- Prove the existence of Glosten-Milgrom equilibrium;
- Remove technical assumptions in the convergence.

Future research:

- When noisy trader can submit multiple orders, then pricing rule can behavior like a LOB:

\[
\sum_{i=1}^{n} a_{i+}(t, z), \quad \text{where} \quad a_{i+}(t, z) = \mathbb{P}^{t, z}(Z_1 \geq y | \Delta Z_t \geq i).
\]

Here \( i \rightarrow a_{i+} \) is increasing.

- Risk averse insider / market maker.
- multiple insiders / market makers.
Thanks for your attention!