Optimal Contracting with Unobservable Managerial Hedging

Yu Huang\textsuperscript{1}  Nengjiu Ju\textsuperscript{1}  Hao Xing\textsuperscript{2}

\textsuperscript{1}Shanghai Advanced Institute of Finance, Shanghai Jiao Tong University

\textsuperscript{2}Department of Statistics, London School of Economics

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Motivation: Relative Performance Evaluation (RPE)

Holmstrom (82): market shocks, which are not affected by managers, should be removed from managerial compensations.

RPE: Investors filter the market return from the firm’s output to compensate managers.
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Agent (Manager)
- $u(c) = -\frac{1}{\gamma} e^{-\gamma c}$
- reservation utility $u(R_0)$
- exert effort $\alpha$ with cost $h(\alpha) = \frac{1}{2} \alpha^2$
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Output

\[ X = I + M, \quad I \sim N(\alpha, \sigma_I), \quad M \sim N(0, \sigma_M), \quad M, N \text{ indep.} \]
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\]

Principal (Investor)
- Risk neutral
- observe \( X \) and \( M \), not \( \alpha \)
- Pay Agent via the following linear contract
\[
\xi = a + bX + cM,
\]
where \( a, b, c \) are constants.
The optimal contract

Agent’s optimization problem:

$$\max_\alpha \mathbb{E} \left[ -\frac{1}{\gamma} \exp \left( -\gamma (\xi - h(\alpha)) \right) \right].$$
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Equivalent to

$$\max_{\alpha} \left\{ b\alpha - h(\alpha) - \frac{\gamma}{2} \left( b^2 \sigma_i^2 + (b + c)^2 \sigma_M^2 \right) \right\}.$$
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$$\alpha^* = b, \quad a = R_0 - \frac{1}{2} b^2 + \frac{\gamma}{2} \left[ b^2 \sigma_i^2 + (b + c)^2 \sigma_M^2 \right].$$
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Principal’s optimization problem:

$$\max_{b,c} \mathbb{E}\left[ \alpha^* - \xi \right].$$
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\max_\alpha \mathbb{E} \left[ -\frac{1}{\gamma} \exp \left( -\gamma (\xi - h(\alpha)) \right) \right].
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\[
\max_{b,c} \mathbb{E} \left[ \alpha^* - \xi \right].
\]

Equivalent to

\[
\max_{b,c} \left\{ \alpha^* - \frac{1}{2} (\alpha^*)^2 - \frac{\gamma}{2} \left[ b^2 \sigma_i^2 + (b + c)^2 \sigma_M^2 \right] \right\}.
\]

Therefore, \( c^* = -b^* \), RPE \( \xi = a + b^* (X - M) \) is the best!
Empirical tests of RPE

Homlstrom (82) assumes that Managers do not hedge.

- Managers sell stocks to diversify Ofek-Yermack (00), trades financial derivatives Bettis-Bizjak-Lemmon (01)
- Cvitanić-Henderson-Lazrak (14): observable hedging
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Is there a negative relationship between compensation and market return?
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Results are mixed:
  ▶ Negative: Antle-Smith (86), Barro-Barro (90), Jensen-Murphy (90), Janakiraman-Lambert-Larcker (92), Aggrwal and Samwick (99)...
  ▶ Positive: Gong-Li-Shin (11), Albuquerque-De Franco-Verdi (13)
  ▶ Jenter-Kanaan (15): CEOs are more likely to be fired when the peers/market perform badly.
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Messages from the empirical literature:

- Difficult to provide incentive when managers hedge
- The market return is not completely filtered out from compensation
- The relationship between CEO turnover and RPE is puzzling
Unobservable Managerial Hedging

Effort

Project

Agent

Private Portfolio Choice

Compensation

Consumption

(Moral hazard I): hidden effort

(Moral hazard II): unobservable saving & hedging: may offset incentives
Unobservable Managerial Hedging

(Moral hazard I): hidden effort
(Moral hazard II): unobservable saving & hedging: may offset incentives
Economic contributions

Our model imposes limited liability restriction for contract compensation. No negative compensation!

- Inefficient liquidation
- Risk-neutral principal is endogenously risk averse
- Principal shares market risk with agent
- Market contract sensitivity can be positive near liquidation boundary

\[
\text{Compensation} \sim dY_t = Z(Y_t) \text{ output } + U(Y_t) \text{ market},
\]

where \( Y \) is called Agent's contract value.

\[
\text{When } Y \text{ is close to liquidation boundary, } U(Y_t) \text{ can be positive.}
\]

\[
\text{market } \downarrow \Rightarrow Y \downarrow \Rightarrow \text{Liquidation probability } \uparrow.
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Main contributions:
- Contract sensitivities are state dependent

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“Impossible Trinity in Contracting”

Risk Aversion

- H-M (87)
- He (11)
- Williams (15)
- S-DT (16)

Liquidation Boundary

- Sannikov (08)

Private Saving

- D-S (06)
- He (09)
“Impossible Trinity in Contracting”

- Use agent’s **certainty equivalence** as the state variable
- Principal’s problem is stochastic control with **regular + singular** controls
A risk-free bond with rate $r$

A market portfolio with return process

$$dR_t = m dt + \sigma dB_t.$$
Model

A risk-free bond with rate $r$

A market portfolio with return process

$$dR_t = m dt + \sigma dB_t.$$  

The output process of the project

$$dX_t = (\mu + A_t) dt + \rho \psi dB_t + \sqrt{1 - \rho^2 \psi} dB_t^\perp,$$

$X$ and $R$ are observable to the principal continuously.
Agent’s problem

Agent’s private wealth process

\[ dS_t = rS_t dt + \pi_t (m - r) dt + \pi_t \sigma dB_t + dl_t - h(A_t) dt - c_t dt, \]

- \( \pi \): monetary value invested in the market
- \( l \): cumulative compensation, nondecreasing (limited liability)
- \( h(A) = \frac{\kappa}{2} A^2 + bA \): monetary cost for agent’s effort \( A \)
- \( c \): private consumption rate
- Admissibility: transversality condition: \( \lim_{T \to \infty} \mathbb{E}[e^{-\delta T} e^{-r \gamma S_T}] = 0. \)
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A CARA agent with \( u(c) = -\frac{1}{\gamma} e^{-\gamma c} \)

Discounting rate \( \bar{\delta} \)

Agent’s outside option:

\[ V_t = \text{ess sup}_{c, \pi} \mathbb{E}_t \left[ \bar{\delta} \int_t^\infty e^{-\bar{\delta}(s-t)} u(c_s) ds \right], \]

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where \( dS_t = rS_t dt + \pi_t (m - r) dt + \pi_t \sigma dB_t - c_t dt. \)
Agent’s problem

\[ u(G_t) = \text{ess sup}_{A,\pi,c} \mathbb{E}_t \left[ \bar{\delta} \int_t^\tau e^{-\delta(s-t)} u(c_t) dt + e^{-\bar{\delta}(\tau-t)} u(rS_\tau - \ell) \right], \]

where

\[ \tau = \inf\{u \geq 0 : G_u \leq rS_u - \ell\}. \]
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- \( G \) is Agent’s certainty equivalence
- The project is liquided when Agent’s certainty equivalence reaches his outside option
Agent’s problem

\[ u(G_t) = \text{ess sup}_{A, \pi, c} \mathbb{E}_t \left[ \frac{1}{\delta} \int_t^\tau e^{-\delta(s-t)} u(c_t) dt + e^{-\delta(\tau-t)} u(rS_\tau - \ell) \right], \]

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Define

\[ G_t = rS_t - \ell + rY_t. \]

Contract’s additional value to the agent is \( Y \), the Agent’s contract value.
Principal’s problem

Principal is risk neutral

Discounting rate $\delta$

Principal’s problem:

$$\sup \mathbb{E} \left[ \delta \int_0^\tau e^{-\delta t} ((\mu + A^*) dt - dl_t) + e^{-\delta \tau} \phi \mu \right].$$

$\phi \in (0, 1]$ is the liquidation discount
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We choose $Y$ as Principal’s unique state variable.

Principal does not know Agent’s private wealth $S$.

CARA utility assumption is essential.
Principal’s problem

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Goal: Find Principal’s optimal contract $I^*$, Agent's optimal effort $A^*$. 
Dynamics of $Y$

Suppose that the dynamics of $Y$ follows

$$dY_t = dH_t + Z_t dX_t + U_t dR_t$$
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$H$ can be determined by the Martingale Principal:
El Karoui-Rouge (00), Hu-Imkeller-Muller (05)

1. $e^{-\delta t} u(G_t) + \delta \int_0^t e^{-\delta s} u(c_s) ds$ is a supermartingale until $\tau$ for arbitrary strategy $A, \pi, c$;
2. it is a martingale for the optimal strategy.
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$$dY_t = \left[ rY_t + \frac{r\gamma}{2} \psi^2 (1 - \rho^2) Z_t^2 + h(A^*(Z_t)) + (m - r)\zeta_t \right] dt - dl_t$$

$$+ \zeta_t \sigma dB_t + Z_t \sqrt{1 - \rho^2} \psi dB_t$$

- $\zeta = \frac{\rho \psi}{\sigma} Z + U$ is the agent's exposure to the market
- Agent's optimal portfolio is $\pi^* = \frac{m-r}{r\gamma \sigma^2} - \zeta$
- $\tau = \inf\{ t \geq 0 : Y_t \leq 0 \}$ is the liquidation time
- $A^*(Z) = \arg \min\{ h(A) - ZA \}$
Consider $Y$ as Principal’s unique state variable

$$
W(y) = \sup_{l, Z, \zeta} \mathbb{E} \left[ \delta \int_0^{\tau} e^{-\delta t} \left( (\mu + A^*(Z_t)) dt - dl_t \right) + e^{-\delta \tau} \mu \right],
$$

where $\tau = \inf \{ t \geq 0 : Y_t \leq 0 \}$

- $Z, \zeta$: regular control, $Z \in [\underline{Z}, \overline{Z}]$
- $l$: singular control
Variational inequality

\[
\min \left\{ \delta W - \sup_{Z, \zeta} \left\{ \delta (\mu + A^*(Z)) + (r y + g(Z, \zeta)) W' \right. \right. \\
+ \frac{1}{2} \left[ \sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 Z^2 \right] W'' \left. \right\}, \\
W' + \delta \right\} = 0,
\]

where

\[
g(Z, \zeta) = \frac{r \gamma}{2} \psi^2 (1 - \rho^2) Z^2 + (m - r) \zeta + h(A^*(Z)).
\]

Cost of hedging

Cost of effort
Variational inequality

\[
\min \left\{ \delta W - \sup_{Z, \zeta} \left\{ \delta (\mu + A^*(Z)) + (ry + g(Z, \zeta)) W' \right\} + 1/2 \left[ \sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 Z^2 \right] W'' \right\},
\]

\[
W' + \delta = 0,
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g(Z, \zeta) = \frac{r \gamma}{2} \psi^2 (1 - \rho^2) Z^2 + (m - r) \zeta + h(A^*(Z)).
\]

A free boundary problem:

\[
\delta W = \sup_{Z, \zeta} \left\{ \delta (\mu + A^*(Z)) + (ry + g(Z, \zeta)) W' + 1/2 \left[ \sigma^2 \zeta^2 + (1 - \rho^2) \psi^2 Z^2 \right] W'' \right\}
\]

\[
W'(\bar{y}) = -\delta, \quad W''(\bar{y}) = 0,
\]

\[
W(0) = \phi \mu.
\]
Main result

Theorem
Assume that

1. \( r > \delta \) (ensure \( \bar{y} \) is finite),
2. \( Z > 0 \) (ensure the HJB is uniform elliptic).

There is a unique solution \( W \in C^2(0, \infty) \) of the variational inequality. Moreover,

1. \( W \) is strictly concave on \((0, \bar{y})\),
2. \( W \) satisfies the free boundary problem,
3. The optimal contract is a “local time” type, which reflects \( Y \) at \( \bar{y} \).
Risk sharing and incentive provision

Panel A: Principal’s Value Function under OPE

Panel B: Optimal Gross Market Exposure

Panel C: Optimal Sensitivity to Output

Panel D: Optimal Sensitivity to Market Return
Economic results

1. $W$ is concave, principal is endogenously risk averse
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2. The optimal exposure to the market is

$$\zeta^* = -\frac{m - r}{\sigma^2} \frac{W'(y)}{W''(y)}.$$ 

When $m > r$,

- When $Y$ is close to the liquidation boundary: $W' > 0 \implies \zeta^* > 0$
- When $Y$ is close to the payment boundary: $W' < 0 \implies \zeta^* < 0$
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3. $U^* = \zeta^* - \frac{\rho \psi}{\sigma} Z^*$ can be positive when $Y$ is close to the liquidation boundary
Economic results

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4.

$$dY_t = dH_t + Z_t dX_t + U_t dR_t.$$  

When $Y$ is close to 0, positive $U$ implies

$$dR_t < 0 \implies Y_t \text{ closer to } 0 \implies \mathbb{P}(\text{liquidation}) \uparrow$$
Conclusion

- A model with unobservable managerial hedging
- Market contract sensitivity is dynamic and can be positive; OuYang (05), Ozdenoren-Yuan (17)
- Risk aversion + private saving/investment + liquidation
- Positive market contract sensitivity implies more liquidation when market falls
Conclusion

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Thanks for your attention!
Model comparison

\[ dY_t = dH_t + Z_t dX_t + U_t dR_t. \]

- APE: \( U \equiv 0 \)
- RPE: \( U = -\frac{\rho \psi}{\sigma} Z \)
- OPE: \( U \) can be chosen freely
- Benchmark: observable hedging, unobservable effort
Proofs

\[
\min \left\{ \delta \mathcal{W} - \sup_{Z, \zeta} \left\{ \delta (\mu + A^*(Z)) + (ry + g(Z, \zeta)) W' + \frac{1}{2} \Sigma(Z, \zeta) W'' \right\}, \right. \\
\left. W' + \delta \right\} = 0
\]

1. \( \underline{\mathcal{W}} \leq \mathcal{W} \leq \overline{\mathcal{W}} \), where \( \overline{\mathcal{W}}(y) = \mu - \delta y + \sup_{Z, \zeta} \{A^*(Z) - g(Z, \zeta)\} \)
and \( \underline{\mathcal{W}}(y) = \phi \mu - \delta y \).

2. \( \mathcal{W} \) is viscosity solution (DPP)

3. unique viscosity solution with linear growth (controls need to be bounded), hence \( \mathcal{W} \) is continuous

4. \( \tilde{\mathcal{W}}(y) = \mu - ry + \sup_{Z, \zeta} \{A^*(Z) - g(Z, \zeta)\} \). The free boundary is before the intersection of \( \underline{\mathcal{W}} \) and \( \tilde{\mathcal{W}} \) (\( r > \delta \))

5. \( \mathcal{W} \) is concave

\[
\mathcal{W}'' = \inf_{Z, \zeta} \left\{ \frac{\delta \mathcal{W} - \delta [\mu + A^*(Z) - (ry + g(Z, \zeta))] W'}{\frac{1}{2} \Sigma(Z, \zeta)} \right\}
\]

6. \( \mathcal{W} \) is \( C^2 \) (uniformly elliptic) Strulovici-Szydlowski (2015), Pham (09)