Portfolio turnpikes in incomplete markets

Hao Xing

London School of Economics

joint work with
Paolo Guasoni, Kostas Kardaras, and Scott Robertson

Stochastic Analysis in Finance and Insurance, Ann Arbor, May 18, 2011
What is a turnpike?
What is a turnpike? Express way
What is the portfolio turnpike?

Merton’s problem:

\[ u^T(x) = \max_{X \text{admissible}} \mathbb{E}[U(X_T)]. \]
What is the portfolio turnpike?

Merton’s problem:

\[ u^T(x) = \max_{X \text{admissible}} \mathbb{E}[U(X_T)]. \]

When \( U(x) = x^p/p \), with \( p < 1 \), and the market is Black-Scholes,

\[ \pi = \frac{1}{1 - p} \sum^{-1} \mu. \]
What is the portfolio turnpike?

Merton’s problem:

\[ u^T(x) = \max_{X \text{admissible}} \mathbb{E}[U(X_T)]. \]

When \( U(x) = x^p / p \), with \( p < 1 \), and the market is Black-Scholes,

\[ \pi = \frac{1}{1 - p} \Sigma^{-1} \mu. \]

For more general utility function and market structure:

**Existence:** Karatzas-Shreve (98), Kramkov-Schachermayer (99)
**Explicit solutions:** Kallsen (00), Zariphopoulou (01), Goll-Kallsen (03), Hu-Imkeller-Muller (05)
What is the portfolio turnpike?

Merton’s problem:

\[ u^T(x) = \max_{X \text{admissible}} \mathbb{E}[U(X_T)]. \]

When \( U(x) = x^p/p \), with \( p < 1 \), and the market is Black-Scholes,

\[ \pi = \frac{1}{1 - p} \Sigma^{-1} \mu. \]

For more general utility function and market structure:

Existence: Karatzas-Shreve (98), Kramkov-Schachermayer (99)

Explicit solutions: Kallsen (00), Zariphopoulou (01), Goll-Kallsen (03), Hu-Imkeller-Muller (05)

Portfolio Turnpike:

When investment horizon is distant and market grows indefinitely,

Optimal portfolios for generic utilities

is close to

Optimal portfolios of power or logarithmic utilities.
What is the portfolio turnpike?

Merton’s problem:

\[ u^T(x) = \max_{X \text{admissible}} \mathbb{E}[U(X_T)]. \]

When \( U(x) = x^p/p \), with \( p < 1 \), and the market is Black-Scholes,

\[ \pi = \frac{1}{1 - p} \sum^{-1} \mu. \]

For more general utility function and market structure:

**Existence:** Karatzas-Shreve (98), Kramkov-Schachermayer (99)

**Explicit solutions:** Kallsen (00), Zariphopoulou (01), Goll-Kallsen (03), Hu-Imkeller-Muller (05)

**Portfolio Turnpike:**

When investment horizon is distant and market grows indefinitely,

Optimal portfolios for generic utilities

is close to

Optimal portfolios of power or logarithmic utilities.

For long term investments, follow the CRRA turnpike!
A more mathematical formulation

Consider problems

\[ u^{0,T} = \sup_{X \text{ admissible}} \mathbb{E}^P[X_T^p/p], \quad u^{1,T} = \sup_{X \text{ admissible}} \mathbb{E}^P[U(X_T)]. \]

Suppose that

- **Distant horizon:** \( T \to \infty. \)
- **Market grows indefinitely:** \( \exists X \text{ s.t. } \lim_{T \to \infty} X_T = \infty. \)
- **Similar utilities at large wealth levels:**
  \[ \lim_{x \uparrow \infty} \frac{U'(x)}{x^{p-1}} = 1. \]
A more mathematical formulation

Consider problems

\[ u^{0,T} = \sup_{X \text{ admissible}} \mathbb{E}^p[X_T^p/p], \quad u^{1,T} = \sup_{X \text{ admissible}} \mathbb{E}^p[U(X_T)]. \]

Suppose that

- Distant horizon: \( T \rightarrow \infty \).
- Market grows indefinitely: \( \exists X \text{ s.t. } \lim_{T \rightarrow \infty} X_T = \infty \).
- Similar utilities at large wealth levels:

\[ \lim_{x \uparrow \infty} \frac{U'(x)}{x^{p-1}} = 1. \]

Then \( \pi_s^{1,T} \) and \( \pi_s^{0,T} \) are similar for \( s \in [0, t] \) with \( t \) is fixed.
Literature

Discrete time:

- Mossin (68): \(-U'(x)/U''(x) = ax + b\)
- Leland (72), Ross (74), Hakansson (74)
- Huberman-Ross (83): \(U'(x)\) is regular varying at \(\infty\):
  \[
  \lim_{x \to \infty} \frac{U'(ax)}{U'(x)} = a^{p-1}, \quad \text{for any} \ a > 0.
  \]

Continuous time:

- Cox-Huang (92)
- Jin (98): include consumptions
- Huang-Zariphopouou (99): viscosity solutions
- Dybvig-Rogers-Back (99): NO independent returns, complete markets with Brownian filtration
Literature

Discrete time:

- Mossin (68): $-U'(x)/U''(x) = ax + b$
- Leland (72), Ross (74), Hakansson (74)
- Huberman-Ross (83): $U'(x)$ is regular varying at $\infty$:
  \[
  \lim_{x \to \infty} \frac{U'(ax)}{U'(x)} = a^{p-1}, \quad \text{for any } a > 0.
  \]

Continuous time:

- Cox-Huang (92)
- Jin (98): include consumptions
- Huang-Zariphopouou (99): viscosity solutions
- Dybvig-Rogers-Back (99): NO independent returns, complete markets with Brownian filtration

Assume independent return, except Dybvig et al. $\implies$ DPP.

Assume market completeness $\implies$ duality method.
Incomplete markets with stochastic investment opportunities

The portfolio choice problem is least tractable, turnpike theorems are most needed.
Incomplete markets with stochastic investment opportunities

The portfolio choice problem is least tractable, turnpike theorems are most needed.

We prove three kind of turnpike theorems:

- **Abstract Turnpike**: minimum market assumptions, converge under myopic probabilities except for the log utility;

- **Classical Turnpike**: a class of diffusion models with stochastic drift and volatility. Convergence is shown under the physical measure.

- **Explicit Turnpike**: finite horizon optimal portfolios for generic utilities converge to the long-run optimal portfolio for power utility.
Admissible wealth processes

For each $T \in \mathbb{R}_+$, wealth process is chosen from a set of nonneg. semimartingales $\mathcal{X}^T$, such that

i) $X_0 = 1$ for all $X \in \mathcal{X}^T$;

ii) $\mathcal{X}^T$ contains some strictly positive $X$;

iii) $\mathcal{X}^T$ is convex;

iv) $\mathcal{X}^T$ is stable under compounding: for any $X, X' \in \mathcal{X}^T$ and a $[0, T]$ valued stopping time $\tau$,

$$X'' = X\mathbb{I}_{[0, \tau]} + X'\frac{X_{\tau}}{X'_{\tau}}\mathbb{I}_{[\tau, T]} \in \mathcal{X}^T.$$  

One can switch to another strategy at any stopping time in a self-financing way.
Admissible wealth processes

For each $T \in \mathbb{R}_+$, wealth process is chosen from a set of nonneg. semimartingales $\mathcal{X}^T$, such that

i) $X_0 = 1$ for all $X \in \mathcal{X}^T$;

ii) $\mathcal{X}^T$ contains some strictly positive $X$;

iii) $\mathcal{X}^T$ is convex;

iv) $\mathcal{X}^T$ is stable under compounding: for any $X, X' \in \mathcal{X}^T$ and a $[0, T]$ valued stopping time $\tau$,

$$X'' = X\mathbb{I}_{[0,\tau]} + X'\frac{X_\tau}{X'_\tau}\mathbb{I}_{[\tau, T]} \in \mathcal{X}^T.$$ 

One can switch to another strategy at any stopping time in a self-financing way.

These assumptions are satisfied in general frictionless market with semimartingale dynamics.
Assumptions on utilities

Let $\mathcal{R}(x) := U'(x)/x^{p-1}$.

\[
\lim_{x \uparrow \infty} \mathcal{R}(x) = 1,
\lim \inf_{x \downarrow 0} \mathcal{R}(x) > 0, \quad \text{for } 0 \neq p < 1,
\lim \sup_{x \downarrow 0} \mathcal{R}(x) < \infty, \quad \text{for } p < 1.
\]
Assumptions on utilities

Let $\mathcal{R}(x) := U'(x)/x^{p-1}$.

\[
\lim_{x \uparrow \infty} \mathcal{R}(x) = 1,
\]
\[
\lim \inf_{x \downarrow 0} \mathcal{R}(x) > 0, \quad \text{for } 0 \neq p < 1,
\]
\[
\lim \sup_{x \downarrow 0} \mathcal{R}(x) < \infty, \quad \text{for } p < 1.
\]

Assumptions in the literature:

1. $\lim_{x \to \infty} \frac{U'(x)}{x^{p-1}} = 1$,
2. $\lim_{x \to \infty} -xU''(x) \frac{U'(x)}{U'(x)} = 1 - p$,
3. $\lim_{x \to \infty} \frac{U'(ax)}{U'(x)} = a^{p-1}$ for any $a > 0$.

1. $\implies$ 2. $\implies$ 3., but not in the other direction.
Assumptions on utilities

Let $\mathcal{R}(x) := \frac{U'(x)}{x^{p-1}}$.

$$\lim_{x \uparrow \infty} \mathcal{R}(x) = 1,$$
$$\lim \inf_{x \downarrow 0} \mathcal{R}(x) > 0, \quad \text{for } 0 \neq p < 1,$$
$$\lim \sup_{x \downarrow 0} \mathcal{R}(x) < \infty, \quad \text{for } p < 1.$$

Assumptions in the literature:

1. $\lim_{x \to \infty} \frac{U'(x)}{x^{p-1}} = 1$,
2. $\lim_{x \to \infty} - \frac{xU''(x)}{U'(x)} = 1 - p$,
3. $\lim_{x \to \infty} \frac{U'(ax)}{U'(x)} = a^{p-1}$ for any $a > 0$.

1. $\implies$ 2. $\implies$ 3., but not in the other direction.

Dybvig-Rogers-Back (99) does not have assumptions on $\mathcal{R}(x)$ at $x = 0$. 
Assumptions on portfolio choice problem

We assume the portfolio choice problem is well-posed for each horizon:

For all \( T > 0 \) and \( i = 0, 1, \)

\[ -\infty < u^{i,T} < \infty; \]

- optimal wealth processes \( X^{i,T} \) exist.

Together with the admissible assumption, this implies the no arbitrage condition:

\[ X_s = 0 \implies X_t = 0, \text{ for any } t \geq s. \]
Myopic probabilities

Define the myopic probabilities \((\mathbb{P}^T)_{T \geq 0}\) as

\[
\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\left(\chi^0, T\right)^p}{\mathbb{E}^\mathbb{P}\left[\left(\chi^0, T\right)^p\right]}.
\]
Myopic probabilities

Define the myopic probabilities \((\mathbb{P}^T)_{T \geq 0}\) as

\[
\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\left(X_0^0, T\right)^p}{\mathbb{E}^\mathbb{P} \left[\left(X_0^0, T\right)^p\right]}.
\]

- Appear in Kramkov-Sirbu (06, 07);
- \(\mathbb{P}^T = \mathbb{P}\) when log utility is considered;
- Power optimal under \(\mathbb{P}\) is log optimal under \(\mathbb{P}^T\).
Myopic probabilities

Define the myopic probabilities \((\mathbb{P}^T)_{T \geq 0}\) as

\[
\frac{d\mathbb{P}^T}{d\mathbb{P}} = \frac{\left( X^0_T, T \right)^p}{\mathbb{E}^\mathbb{P} \left[ \left( X^0_T, T \right)^p \right]}.
\]

- Appear in Kramkov-Sirbu (06, 07);
- \(\mathbb{P}^T = \mathbb{P}\) when log utility is considered;
- Power optimal under \(\mathbb{P}\) is log optimal under \(\mathbb{P}^T\).

First order condition

\[
\mathbb{E}^\mathbb{P} \left[ \left( X^0_T, T \right)^{p-1} (X_T - X^0_T) \right] \leq 0 \text{ for any } X \in \mathcal{X}^T.
\]

Change the measure to \(\mathbb{P}^T\),

\[
\mathbb{E}^{\mathbb{P}^T} \left[ X_T / X^0_T \right] \leq 1, \quad \text{for any } X \in \mathcal{X}^T.
\]

Hence \(X^0_T, T\) has numéraire property under \(\mathbb{P}^T\).
Last assumption

In the Black-Scholes case:

\[
\frac{d\mathbb{P}^T}{d\mathbb{P}} = \xi_T X_T^0,
\]

where \( \xi \) is the density of the unique martingale measure.

Note:

\[
\lim_{T \to \infty} \frac{d\mathbb{P}^T}{d\mathbb{P}} = 0, \quad \mathbb{P} - a.s. \quad \text{if } p\lambda \neq 0.
\]
Last assumption
In the Black-Scholes case:
\[ \frac{d\mathbb{P}^T}{d\mathbb{P}} = \xi_T X_T^0, \]
where $\xi$ is the density of the unique martingale measure.

Note:
\[ \lim_{T \to \infty} \frac{d\mathbb{P}^T}{d\mathbb{P}} = 0, \quad \mathbb{P} - a.s. \quad \text{if } p\lambda \neq 0. \]

Assumption on market growth:
There exists $(\hat{X}^T)_{T \geq 0}$ such that $\hat{X}^T \in \mathcal{X}^T$ and
\[ \lim_{T \to \infty} \mathbb{P}^T(\hat{X}^T_T \geq N) = 1, \quad \text{for any } N > 0. \]
Last assumption
In the Black-Scholes case:

\[ \frac{d\mathbb{P}^T}{d\mathbb{P}} = \xi_T X_T^0, \]

where \( \xi \) is the density of the unique martingale measure.

Note:

\[ \lim_{T \to \infty} \frac{d\mathbb{P}^T}{d\mathbb{P}} = 0, \quad \mathbb{P} - a.s. \quad \text{if } p\lambda \neq 0. \]

Assumption on market growth:
There exists \((\hat{X}^T)_{T \geq 0}\) such that \(\hat{X}^T \in \mathcal{X}^T\) and

\[ \lim_{T \to \infty} \mathbb{P}^T (\hat{X}_T^T \geq N) = 1, \quad \text{for any } N > 0. \]

This condition is satisfied in

- Safe rate bounded from below by a constant \( r > 0 \).
- Black-Scholes market \( dS_u/S_u = \mu du + \sigma dW_u \),
  \( \hat{X} \) can be chosen as the power optimal \( X^{0,T} \) when \( \mu \neq 0 \).
First turnpike theorem

Optimal terminal wealths are unbounded as $T \rightarrow \infty$.

Define

$$r_u^T := \frac{X_u^{1,T}}{X_u^{0,T}}, \quad \Pi_u^T := \int_0^u dr_v^T / r_v^T, \quad u \in [0, T].$$
First turnpike theorem

Optimal terminal wealths are unbounded as $T \to \infty$.

Define

\[ r^T_u := X^{1,T}_u / X^{0,T}_u, \quad \Pi^T_u := \int_0^u dr^T_v / r^T_v, \quad u \in [0, T]. \]

Theorem (Abstract Turnpike)

Under above assumptions, for any $\epsilon > 0$,

a) $\lim_{T \to \infty} \mathbb{P}^T \left( \sup_{u \in [0, T]} |r^T_u - 1| \geq \epsilon \right) = 0,$

b) $\lim_{T \to \infty} \mathbb{P}^T \left( [\Pi^T, \Pi^T]_T \geq \epsilon \right) = 0.$

Remark:

- In the market $dS_u / S_u = \mu_u du + \sigma_u dW_u$,

\[ [\Pi^T, \Pi^T]_T. = \int_0^T \left| (\pi^{1,T}_u - \pi^{0,T}_u) \sigma_u \right|^2 du. \]

- When log utility is considered, the convergence is under $\mathbb{P}$. 
An Example

In Huberman-Ross (83), a discrete time market model with independent return was studied.

$U$ is regular varying at $\infty$ is a necessary condition for turnpike.

Consider a Black-Scholes model with $r = 0$ and $\frac{dS_t}{S_t} = \mu dt + \sigma dW_t$.

Let $X$ be the log-optimal wealth process and $\tau_n = \inf\{t \geq 0 | X_t = n\}$.

For problem with horizon $\tau_n$, $\xi$ is the optimal wealth process for generic utility $U$.

$$E[\xi_{\tau_n}/X_{\tau_n}] \leq 1 \Rightarrow E[\xi_{\tau_n}] \leq n,$$

for any wealth process $\xi$.

$$E[U(\xi_{\tau_n})] \leq U(E[\xi_{\tau_n}]) \leq U(n) = E[U(X_{\tau_n})].$$

Then change time from $\tau_n$ to $n$, the return becomes dependent.

Turnpike trivially holds. But the generic utility may not be regularly varying.
An Example

In Huberman-Ross (83), a discrete time market model with independent return was studied.

$U$ is regular varying at $\infty$ is a necessary condition for turnpike.

In continuous time with dependent return, this does not hold.

Consider a Black-Scholes model with $r = 0$ and $dS_t/S_t = \mu \, dt + \sigma \, dW_t$.

Let $X$ be the log-optimal wealth process and $\tau_n = \inf\{t \geq 0 \mid X_t = n\}$.

For problem with horizon $\tau_n$, $\xi$ is the optimal wealth process for generic utility $U$.

$$\mathbb{E}[\xi_{\tau_n} / X_{\tau_n}] \leq 1 \quad \Longrightarrow \quad \mathbb{E}[\xi_{\tau_n}] \leq n, \quad \text{for any wealth process } \xi.$$

$$\mathbb{E}[U(\xi_{\tau_n})] \leq U(\mathbb{E}[\xi_{\tau_n}]) \leq U(n) = \mathbb{E}[U(X_{\tau_n})].$$

Then change time from $\tau_n$ to $n$, the return becomes dependent.

Turnpike trivially holds. But the generic utility may not be regularly varying.
Drawback

Convergence is under $\{P^T\}_{T \geq 0}$,

Limit of $(\mathbb{P}^T)_{T \geq 0}$ is in general singular wrt $\mathbb{P}$. 
Drawback

Convergence is under $\{P^T\}_{T \geq 0}$.

Limit of $(\mathbb{P}^T)_{T \geq 0}$ is in general singular wrt $\mathbb{P}$.

So we look at a time window $[0, t]$

$$\lim_{T \to \infty} \mathbb{P}^T \left( \sup_{u \in [0, t]} |r_u^T - 1| \geq \epsilon \right) = 0,$$

$$\lim_{T \to \infty} \mathbb{P}^T \left( \left[ \Pi^T, \Pi^T \right]_t \geq \epsilon \right) = 0.$$
Drawback

Convergence is under \( \{ P^T \}_{T \geq 0} \),

Limit of \( (P^T)_{T \geq 0} \) is in general singular wrt \( P \).

So we look at a time window \([0, t]\)

\[
\lim_{T \to \infty} P^T \left( \sup_{u \in [0, t]} \left| r_u^T - 1 \right| \geq \epsilon \right) = 0,
\]

\[
\lim_{T \to \infty} P^T \left( \left[ \Pi^T, \Pi^T \right]_t \geq \epsilon \right) = 0.
\]

Then consider the conditional densities

\[
\frac{dP^T}{dP} \bigg|_{\mathcal{F}_t} .
\]

Question:

- Does \( \lim_{T \to \infty} \frac{dP^T}{dP} \big|_{\mathcal{F}_t} \) exist?
- Does the limit induce a measure equivalent to \( P \) on \( \mathcal{F}_t \)?
Corollary

1. $X_{t}^{0,T} = X_{t}^{0,S} \equiv X_{t}$ a.s. for all $t \leq S, T$ (myopic optimality);
2. $X_{T}^{0,T} / X_{t}^{0,T}$ and $\mathcal{F}_{t}$ are independent for all $t \leq T$ (independent returns).

then, for any $\epsilon > 0$ and $t \geq 0$:

a) $\lim_{T \to \infty} \mathbb{P} \left( \sup_{u \in [0,t]} |r^{T}_{u} - 1| \geq \epsilon \right) = 0,$

b) $\lim_{T \to \infty} \mathbb{P} \left( [\prod_{T}^{T}, \prod_{T}^{T}]_{t} \geq \epsilon \right) = 0.$
Corollary

1. \( X_{t}^{0,T} = X_{t}^{0,S} \equiv X_{t} \text{ a.s. for all } t \leq S, T \) (myopic optimality);

2. \( X_{t}^{0,T} / X_{t}^{0,T} \text{ and } \mathcal{F}_{t} \text{ are independent for all } t \leq T \) (independent returns).

then, for any \( \epsilon > 0 \) and \( t \geq 0 \):

a) \( \lim_{T \to \infty} \mathbb{P} \left( \sup_{u \in [0,t]} \left| r_{u}^{T} - 1 \right| \geq \epsilon \right) = 0 \),

b) \( \lim_{T \to \infty} \mathbb{P} \left( \left[ \prod_{T}^{T}, \prod_{T}^{T} \right]_{t} \geq \epsilon \right) = 0 \).

Remark: This contains exponential Lévy markets, Kallsen (00).
Markets with stochastic investment opportunities

Consider a diffusion model, investment opportunities are driven by

\[ dY_t = b(Y_t)dt + a(Y_t)dW_t, \]

whose state space is \( E = (\alpha, \beta) \), with \(-\infty \leq \alpha < \beta \leq \infty\).

The market includes a safe rate \( r(Y_t) \) and \( d \) risky assets

\[ \frac{dS^i_t}{S^i_t} = r(Y_t)dt + dR^i_t, \] with
\[ dR_t = \mu(Y_t)dt + \sigma'(Y_t)dZ_t, \]
\[ d\langle Z^i, W \rangle_t = \rho^i(Y_t)dt. \]

Define

\[ A = a^2, \quad \Sigma = \sigma \sigma', \quad \Upsilon = \sigma \rho a. \]
Markets with stochastic investment opportunities

Consider a diffusion model, investment opportunities are driven by

$$dY_t = b(Y_t)dt + a(Y_t)dW_t,$$

whose state space is $E = (\alpha, \beta)$, with $-\infty \leq \alpha < \beta \leq \infty$.

The market includes a safe rate $r(Y_t)$ and $d$ risky assets

$$\frac{dS^i_t}{S^i_t} = r(Y_t)dt + dR^i_t, \quad \text{with}$$

$$dR_t = \mu(Y_t)dt + \sigma'(Y_t)dZ_t,$$

$$d\langle Z^i, W \rangle_t = \rho^i(Y_t)dt.$$

Define

$$A = a^2, \quad \Sigma = \sigma \sigma', \quad \text{and} \quad \Upsilon = \sigma \rho a.$$

We assume the martingale problem associated to $(Y, R)$ is well posed.

Assumption on correlation: $\rho' \rho$ is a constant.
Finite horizon problem for power

Zariphopoulou (01) introduced the power transform. The nonlinear HJB equation is linearized.

\[ u^T(x, y, t) = \frac{x^p}{p} \left( v^T (y, t) \right)^\delta. \]

\( v^T \) solves the reduced HJB equation:

\[ \partial_t v + \mathcal{L} v + cv = 0, \quad v(T, y) = 1, \]

where \( \mathcal{L} = \frac{1}{2} A \partial_{yy}^2 + (b - q \gamma \Sigma^{-1} \mu) \partial_y \) and \( c = \frac{1}{\delta} (pr - \frac{q}{2} \mu' \Sigma^{-1} \mu). \)

The optimal portfolio is

\[ \pi^T = \frac{1}{1 - p} \Sigma^{-1} \left( \mu + \delta \gamma \frac{v_y^T}{v^T} \right) \text{ evaluated on } (t, Y_t). \]
Finite horizon problem for power

Zariphopoulou (01) introduced the power transform. The nonlinear HJB equation is linearized.

\[ u^T(x, y, t) = \frac{x^p}{p} \left( v^T(y, t) \right)^{\delta}. \]

\( v^T \) solves the reduced HJB equation:

\[ \partial_t v + \mathcal{L} v + cv = 0, \quad v(T, y) = 1, \]

where \( \mathcal{L} = \frac{1}{2} A \partial_{yy}^2 + (b - q\gamma \Sigma^{-1} \mu) \partial_y \) and \( c = \frac{1}{\delta} (pr - \frac{q}{2} \mu' \Sigma^{-1} \mu) \).

The optimal portfolio is

\[ \pi^T = \frac{1}{1 - p} \Sigma^{-1} \left( \mu + \delta \gamma \frac{v_y^T}{v^T} \right) \text{ evaluated on } (t, Y_t). \]

Under \( \mathbb{P}^T \), the dynamics of \((R, Y)\) is

\[
\begin{cases}
    dR_t = \frac{1}{1-p} \left( \mu + \delta \gamma \frac{v_y^T}{v^T} \right) (t, Y_t) \, dt + \sigma(Y_t) \, d\tilde{Z}_t \\
    dY_t = \left( B + A \frac{v_y^T}{v^T} \right) (t, Y_t) \, dt + a(Y_t) \, d\tilde{W}_t
\end{cases}
\]
Long-run measure

Our goal: study the limit behavior of \( \left( \frac{d\mathbb{P}^T}{d\mathbb{P}^t}\big|\mathcal{F}_t \right)_{T \geq 0} \).

Guasoni-Robertson (10) proposed that the long-run optimal strategy is

\[ \hat{\pi} = \frac{1}{1 - p \Sigma^{-1} (\mathbf{\mu} + \delta \mathbf{\Upsilon} \hat{v} \mathbf{y} \hat{v})} \]

\( \hat{\pi} \) is the log optimal strategy under \( \hat{\mathbb{P}} \).

Then \( d\hat{\mathbb{P}}/d\mathbb{P}^t \big|\mathcal{F}_t \) is the natural candidate for \( \lim_{T \to \infty} \frac{d\mathbb{P}^T}{d\mathbb{P}^t} \big|\mathcal{F}_t \).
Long-run measure

Our goal: study the limit behavior of \( (d\mathbb{P}^T/d\mathbb{P}|\mathcal{F}_t)_{T \geq 0} \).

Consider the ergodic HJB equation:

\[
\mathcal{L}\hat{v} + c\hat{v} = \lambda\hat{v}.
\]

This relates to the risk sensitive control

\[
\max_{\hat{\pi}} \lim_{T \to \infty} \frac{1}{pT} \log (\mathbb{E} [U(X_T)]) .
\]
Long-run measure

Our goal: study the limit behavior of \( \left( \frac{d\mathbb{P}^T}{d\mathbb{P}|\mathcal{F}_t} \right)_{T \geq 0} \).

Consider the ergodic HJB equation:

\[
\mathcal{L}\hat{\nu} + c\hat{\nu} = \lambda\hat{\nu}.
\]

This relates to the risk sensitive control

\[
\max_{\pi} \lim_{T \to \infty} \frac{1}{pT} \log (\mathbb{E}[U(X_T)]) .
\]

Define \( \hat{\mathbb{P}} \) as the solution to the following martingale problem:

\[
\begin{align*}
    dR_t &= \frac{1}{1-p} \left( \mu + \delta \gamma \frac{\hat{\nu}_y}{\hat{\nu}} \right) (Y_t) \, dt + \sigma \, d\hat{Z}_t \\
    dY_t &= \left( B + A \frac{\hat{\nu}_y}{\hat{\nu}} \right) (Y_t) \, dt + a \, d\hat{W}_t
\end{align*}
\]

Guasoni-Robertson (10) proposed that the long-run optimal strategy is

\[
\hat{\pi} = \frac{1}{1-p} \Sigma^{-1} \left( \mu + \delta \gamma \frac{\hat{\nu}_y}{\hat{\nu}} \right) .
\]
Long-run measure

Our goal: study the limit behavior of \( \left( \frac{d\mathbb{P}^T}{d\mathbb{P}|\mathcal{F}_t} \right) \) \( T \geq 0 \).  

Consider the ergodic HJB equation:

\[
\mathcal{L}\hat{v} + c\hat{v} = \lambda\hat{v}.
\]

This relates to the risk sensitive control

\[
\max_{\pi} \lim_{T \to \infty} \frac{1}{pT} \log \left( \mathbb{E} \left[ U(X_T) \right] \right).
\]

Define \( \hat{\mathbb{P}} \) as the solution to the following martingale problem:

\[
\begin{cases} 
    dR_t = \frac{1}{1-p} \left( \mu + \delta \gamma \frac{\hat{v}_y}{\hat{v}} \right) (Y_t) \ dt + \sigma \ d\hat{Z}_t \\
    dY_t = \left( B + A \frac{\hat{v}_y}{\hat{v}} \right) (Y_t) \ dt + a \ d\hat{W}_t
\end{cases}
\]

Guasoni-Robertson (10) proposed that the long-run optimal strategy is

\[
\hat{\pi} = \frac{1}{1-p} \Sigma^{-1} \left( \mu + \delta \gamma \frac{\hat{v}_y}{\hat{v}} \right).
\]

\( \hat{\pi} \) is the log optimal strategy under \( \hat{\mathbb{P}} \).  

Then \( \frac{d\hat{\mathbb{P}}}{d\mathbb{P}|\mathcal{F}_t} \) is the natural candidate for \( \lim_{T \to \infty} \frac{d\mathbb{P}^T}{d\mathbb{P}|\mathcal{F}_t} \).
Classical Turnpike

Proposition

*When Y is positively recurrent under \( \hat{P} \), then \( \hat{P} \sim P \) on \( F_t \). Moreover*

\[
\hat{P} - \lim_{T \to \infty} \frac{dP^T}{d\hat{P}} |_{F_t} = 1.
\]
Classical Turnpike

Proposition
When $Y$ is positively recurrent under $\hat{P}$, then $\hat{P} \sim P$ on $F_t$. Moreover

$$\hat{P} - \lim_{T \to \infty} \frac{dP^T}{d\hat{P}}|_{F_t} = 1.$$  

This allows to replace $P^T$ in the abstract turnpike with $\hat{P}$.

Theorem (Classical Turnpike)
Under above assumptions, for $0 \neq p < 1$ and any $\epsilon, t > 0$:

a) $\lim_{T \to \infty} P \left( \sup_{u \in [0,t]} |r_u^T - 1| \geq \epsilon \right) = 0$,

b) $\lim_{T \to \infty} P \left( [\Pi^T, \Pi^T]_t \geq \epsilon \right) = 0$.  

Explicit Turnpike

Compare finite horizon optimal portfolio for a generic utility with the long-run optimal for the power utility.

Define
\[
\hat{r}_u^T = \frac{X_u^{1,T}}{\hat{X}_u}, \quad \hat{\Pi}_u^T = \int_0^u \frac{d\hat{r}_v^T}{\hat{r}_v^-}, \quad \text{for } u \in [0, T].
\]

**Theorem (Explicit Turnpike)**

*Under above assumptions, for any \( \epsilon, t > 0 \) and \( 0 \neq p < 1 \):

a) \( \lim_{T \to \infty} \mathbb{P} \left( \sup_{u \in [0,t]} \left| \hat{r}_u^T - 1 \right| \geq \epsilon \right) = 0, \)

b) \( \lim_{T \to \infty} \mathbb{P} \left( \left[ \hat{\Pi}_T, \hat{\Pi}_T \right]_t \geq \epsilon \right) = 0. \)
Explicit Turnpike

Compare finite horizon optimal portfolio for a generic utility with the long-run optimal for the power utility.

Define

\[ \hat{r}_u^T = \frac{X_u^{1,T}}{\hat{X}_u}, \quad \hat{\Pi}_u^T = \int_0^u \frac{d\hat{r}_v^T}{\hat{r}_v^T}, \quad \text{for } u \in [0, T]. \]

**Theorem (Explicit Turnpike)**

*Under above assumptions, for any \( \epsilon, t > 0 \) and \( 0 \neq p < 1 \):

a) \( \lim_{T \to \infty} \mathbb{P} \left( \sup_{u \in [0,t]} \left| \hat{r}_u^T - 1 \right| \geq \epsilon \right) = 0, \)

b) \( \lim_{T \to \infty} \mathbb{P} \left( \left[ \hat{\Pi}_u^T, \hat{\Pi}_v^T \right]_t \geq \epsilon \right) = 0. \)

**Remark:**

- Finite horizon optimal for a generic utility \( \rightarrow \) long-run optimal for power utility;
- The long-run portfolio \( \hat{\Pi} \) is a myopic strategy, expressed by the solution of the ergodic HJB.
Proof for the log case

The first order condition:

\[ \mathbb{E}[U'(X^T)(X_T - X^T)] \leq 0, \quad \text{for any admissible } X. \]
Proof for the log case

The first order condition:

\[ \mathbb{E}[U'(X_T^T)(X_T - X_T^T)] \leq 0, \quad \text{for any admissible } X. \]

Applying the first order condition to log \( x \) and \( U \) respectively.

\[ \mathbb{E}^p[r^T - 1] \leq 0, \]
\[ \mathbb{E}^p[U'(X_1^T)(X_0^T - X_1^T)] \leq 0. \]
Proof for the log case

The first order condition:

$$\mathbb{E}[U'(X_T^T)(X_T - X_T^T)] \leq 0,$$

for any admissible $X$.

Applying the first order condition to $\log x$ and $U$ respectively.

$$\mathbb{E}^P[r_T^T - 1] \leq 0,$$

$$\mathbb{E}^P[U'(X_1^T, T)(X_0^T - X_1^T)] \leq 0.$$

Sum previous two inequalities

$$\mathbb{E}^P \left[ \left( 1 - \frac{R(X_1^T, T)}{r_T^T} \right)(r_T^T - 1) \right] \leq 0.$$

Lemma

$$\mathbb{E}^P \left[ \left\| \left( 1 - \frac{R(X_1^T, T)}{r_T^T} \right)(r_T^T - 1) \right\| \right] \leq 2 \mathbb{E}^P \left[ \left( R(X_1^T, T) - 1 \right)^2 \right].$$

$R$ is bounded, $\mathbb{P}\text{-}\lim_{T \to \infty} R(X_1^T, T) = 1 \implies \mathbb{P} - \lim_{T \to \infty} r_T^T = 1 \implies \mathbb{L}^1 - \lim_{T \to \infty} r_T^T = 1.$
Convergence of portfolios

**Lemma (Kardaras (10))**

Consider a sequence \( \{Y^T\}_{T \geq 0} \) of càdlàg processes and a sequence \( \{\mathbb{P}^T\}_{T \geq 0} \), such that:

i) For each \( T \geq 0 \), \( Y_0^T = 1 \) and \( Y^T \) is strictly positive.

ii) Each \( Y^T \) is a \( \mathbb{P}^T \)-supermartingale.

iii) \( \lim_{T \to \infty} \mathbb{E}^{\mathbb{P}^T}[|Y^T_T - 1|] = 0 \).

Then, for any \( \epsilon > 0 \),

a) \( \lim_{T \to \infty} \mathbb{P}^T \left( \sup_{u \in [0, T]} |Y_u^T - 1| \geq \epsilon \right) = 0 \).

b) \( \lim_{T \to \infty} \mathbb{P}^T (\{L^T_T, L^T_T \geq \epsilon\}) = 0 \), where \( Y^T \) is the stochastic logarithm of \( Y^T \).
Convergence of portfolios

Lemma (Kardaras (10))

Consider a sequence \( \{Y^T\}_{T \geq 0} \) of càdlàg processes and a sequence \( \{\mathbb{P}^T\}_{T \geq 0} \), such that:

i) For each \( T \geq 0 \), \( Y_0^T = 1 \) and \( Y^T \) is strictly positive.

ii) Each \( Y^T \) is a \( \mathbb{P}^T \)-supermartingale.

iii) \( \lim_{T \to \infty} \mathbb{E}^{\mathbb{P}^T}[|Y^T_T - 1|] = 0. \)

Then, for any \( \epsilon > 0 \),

a) \( \lim_{T \to \infty} \mathbb{P}^T(\sup_{u \in [0, T]} |Y^T_u - 1| \geq \epsilon) = 0. \)

b) \( \lim_{T \to \infty} \mathbb{P}^T([L^T, L^T]_T \geq \epsilon) = 0 \), where \( Y^T \) is the stochastic logarithm of \( Y^T \).

Note that \( r^T \) is a supermartingale and \( \mathbb{E}^{\mathbb{P}}[|r^T_T - 1|] = 0 \),

the abstract turnpike follows from the above lemma.
Proof for the diffusion case

Define
\[ h^T(t, y) := \frac{v^T(t, y)}{e^{\lambda c(T-t)\hat{v}(y)}}. \]

It satisfies
\[ \partial_t h^T + \mathcal{L} h^T = 0, \quad (t, y) \in (0, T) \times E, \]
\[ h^T(T, y) = (\hat{v})^{-1}(y), \quad y \in E, \]

where
\[ \mathcal{L} := \mathcal{L} + A \frac{\hat{v}_y}{\hat{v}} \partial_y. \]
Proof for the diffusion case

Define
\[ h^T(t, y) := \frac{v^T(t, y)}{e^{\lambda c(T-t)}\hat{v}(y)}. \]

It satisfies
\[ \partial_t h^T + \hat{\mathcal{L}} h^T = 0, \quad (t, y) \in (0, T) \times E, \]
\[ h^T(T, y) = (\hat{v})^{-1}(y), \quad y \in E, \]

where
\[ \hat{\mathcal{L}} := \mathcal{L} + A\frac{\hat{v}_y}{\hat{v}} \partial_y. \]

This is
- Doob’s h-transform using \( \hat{v} \),
- the generator of \( Y \) under \( \hat{P} \).
Proof for the diffusion case

Define

\[ h_T(t, y) := \frac{v^T(t, y)}{e^{\lambda c(T-t)}\hat{v}(y)}. \]

It satisfies

\[ \partial_t h_T + L h_T = 0, \quad (t, y) \in (0, T) \times E, \]

\[ h_T(T, y) = (\hat{v})^{-1}(y), \quad y \in E, \]

where

\[ \hat{L} := \mathcal{L} + A \frac{\hat{v}_y}{\hat{v}} \partial_y. \]

This is

- Doob’s h-transform using \( \hat{v} \),
- the generator of \( Y \) under \( \hat{P} \).

It has the stochastic representation

\[ h_T(t, y) = \mathbb{E}_{t}^{\hat{P}} Y_T \left[ (\hat{v}(Y_T))^{-1} \right]. \]
Proof for the diffusion case

Define

\[ h^T(t, y) := \frac{\nu^T(t, y)}{e^{\lambda c(T-t)}\hat{\nu}(y)}. \]

It satisfies

\[ \partial_t h^T + \hat{L} h^T = 0, \quad (t, y) \in (0, T) \times E, \]

\[ h^T(T, y) = (\hat{\nu})^{-1}(y), \quad y \in E, \]

where

\[ \hat{L} := \mathcal{L} + A \frac{\hat{\nu}_y}{\hat{\nu}} \partial_y. \]

This is

- Doob’s h-transform using \( \hat{\nu} \),
- the generator of \( Y \) under \( \hat{P} \).

It has the stochastic representation \( h^T(t, y) = \mathbb{E}_{t}^{\hat{P}_y} [(\hat{\nu}(Y_T))^{-1}] \).

\[ \nu^T(t, y) = e^{\lambda c(T-t)}\hat{\nu}(y)\mathbb{E}_{t}^{\hat{P}_y} [(\hat{\nu}(Y_T))^{-1}] = \mathbb{E}_{t}^{\tilde{P}_y} \left[ \exp \left( \int_{t}^{T} c(Y_s)ds \right) \right]. \]

This confirms a remark in Zariphopoulou(01) via h-transform.
Convergence of densities

\[ \frac{d\mathbb{P}^T}{d\hat{\mathbb{P}}} \bigg|_{\mathcal{F}_t} = \frac{h^T(t, Y_t)}{h^T(0, y)} \ldots . \]
Convergence of densities

\[
\frac{d\mathbb{P}^T}{d\hat{\mathbb{P}}} \bigg|_{\mathcal{F}_t} = \frac{h^T(t, Y_t)}{h^T(0, y)} \ldots.
\]

When \( Y \) is positive ergodic under \( \hat{\mathbb{P}} \),

(this can be characterized via the scale function of \( Y \))

\[
\lim_{T \to \infty} h^T(t, y) = \lim_{T \to \infty} \mathbb{E}_t^{\hat{\mathbb{P}}_Y} (\hat{\psi} (Y_T)^{-1}) = K, \quad \text{for all } (t, y).
\]
Verification

1. Assume $\mathcal{L}\hat{v} + c\hat{v} = \lambda\hat{v}$ has a strictly positive $C^2$ solution $\hat{v}$.

2. Define $h^T(t, y) = \mathbb{E}_t^{\hat{P}_y} \left[ (\hat{v}(Y_T))^{-1} \right]$.

3. Show $h^T$ is a $C^{1,2}$ solution to the associated PDE.
   No uniform ellipticity assumption.

4. Define $v^T(t, y) = e^{\lambda c(T-t)}\hat{v}(y)h^T(t, y)$.

5. Verify that $u^T(t, x, y) = \frac{1}{p} x^p \left( v^T(t, y) \right)^\delta$.

   
   $u^T(t, x, y) = \frac{1}{p} x^p \left( v^T(t, y) \right)^\delta \mathbb{E} \left[ \mathcal{E}^T_t \right]$.

   Show $\mathcal{E}^T$ is a martingale.

   We use the result in Cheriditio-Filipovic-Yor (05).
Conclusion and future works

- We show three turnpike theorems in incomplete markets with stochastic investment opportunities.
- The goal is to provide a tractable portfolio for a generic utility when the horizon is long.
- This can be viewed as a stability result for portfolio choice problem when the horizon is long.

Future works:
- Diffusion model with multiple state variables?
- Market with transaction cost?
- Forward utility?
Conclusion and future works

We show three turnpike theorems in incomplete markets with stochastic investment opportunities.

The goal is to provide a tractable portfolio for a generic utility when the horizon is long.

This can be viewed as a stability result for portfolio choice problem when the horizon is long.

Future works:

- Diffusion model with multiple state variables?
- Market with transaction cost?
- Forward utility?
Thanks for your attention!