Valuation equations for stochastic volatility models

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joint work with
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Local martingales in finance

Throughout we assume \( r \equiv 0 \).

**FTAP-I:** NFLVR \iff \exists \ Q \sim \ P, \ s.t. \ S \text{ is a } Q \text{ local martingale.} \]

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Local martingale:

$\exists \{\sigma^n\}_{n \in \mathbb{N}} \uparrow \infty$ such that $\{S_{\sigma^n \wedge t} | t \geq 0\}$ is a martingale for any $n$.

Strict local martingale: local martingale that is not a martingale.
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The loss of martingale property relates to stock price bubble.

[Heston-Loewenstein-Willard 07], [Cox-Hobson 05], [Jarrow-Protter-Shimbo 07, 10].
Strict local martingales in stochastic volatility models

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Example (Andersen-Piterbarg 07)
Consider \( dS_t = S(t)\sqrt{Y_t}dW_t \) and \( dY_t = (\theta - Y_t)dt + Y_t^p dB_t \).

- When \( p \leq 1/2 \), \( S \) is a martingale.
- When \( 1/2 < p \leq 1 \), \( S \) is a martingale if and only if \( \rho \leq 0 \).
Stochastic volatility models

Let us consider

\[ dS_t = S_t \, b(Y_t) \, dW_t, \quad S_0 = x > 0, \]
\[ dY_t = \mu(Y_t) \, dt + \sigma(Y_t) \, dB_t, \quad Y_0 = y > 0, \]

in which \( W \) and \( B \) have constant correlation \( \rho \in (-1, 1) \).

(i) \( \mu : \mathbb{R}_+ \to \mathbb{R} \) satisfies \( \mu(0) \geq 0 \), \( \sigma, b : \mathbb{R}_+ \to \mathbb{R}_+ \) are strictly positive on \( \mathbb{R}_{++} \) and \( \sigma(0) = b(0) = 0 \).

(ii) \( \mu, \sigma^2, b^2, b\sigma \in C^{1,\alpha}(\mathbb{R}_+) \).

(iii) \( \mu \) and \( \sigma \) have at most linear growth, \( (b^2)' \) has at most polynomial growth.
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These assumptions hold for most stochastic volatility models.

This framework allows various model behavior:

- Stock price can be a strict local martingale.
- Volatility process can potentially reach zero.
The valuation equation

Given a European option with the payoff \( g : \mathbb{R}_+ \rightarrow \mathbb{R}_+ \) which is of at most linear growth.

\[
u(x, y, T) := \mathbb{E}[g(S_T) \mid S_0 = x, Y_0 = y].\]
The valuation equation

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A formal application of Itô’s formula gives

\[
\partial_T \nu(x, y, T) = \mathcal{L} \nu(x, y, T), \quad (x, y, T) \in \mathbb{R}_+^3, \\
\nu(x, y, 0) = g(x), \quad (x, y) \in \mathbb{R}_+^2,
\]

in which \( \mathcal{L} := \mu(y) \partial_y + \frac{1}{2} b^2(y)x^2 \partial_{xx} + \frac{1}{2} \sigma^2(y) \partial_{yy} + \rho b(y) \sigma(y) x \partial_{xy} \).
Existence and uniqueness

We want to answer

(Q1) What is the concept of a solution (smoothness and boundary conditions), s.t. \( u \) is one such solution?

(Q2) What is a natural condition under which uniqueness holds in a certain class of functions (of at most linear growth)?
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These questions have been discussed in

[Daskalopoulos-Hamilton 98]
[Heath-Schweizer 00]
[Ekstrom-Tysk 10]
[Constantini-D’ippoliti-Papi 10]
Difficulties

1. Coefficients degenerate at boundaries.
   The standard form of Feynman-Kac formula cannot be applied directly.

[Amadori 07] and [Costantini et al. 10]:
- Underlying process doesn’t reach boundaries.
- Find sufficient conditions for uniqueness.
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2. Non-standard in the PDE literature.

3. Standard viscosity solution techniques cannot be directly applied.
   Coefficients are not Lipschitz continuous in the state space.

[Amadori 07] and [Costantini et al. 10]:
   - Underlying process doesn’t reaches boundaries.
   - Find sufficient conditions for uniqueness.
Multiple solutions

\[ S \text{ is a strict local martingale,} \]
\[ \Downarrow \]
\[ (\text{BS-PDE}) \text{ with boundary conditions has multiple solutions!} \]
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\[ S \text{ is a strict local martingale}, \]

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(BS-PDE) with boundary conditions has \textbf{multiple solutions}!

Consider \( g(x) \equiv 0 \). Define

\[ \delta(x, y, T) := x - \mathbb{E}[S_T | S_0 = x, Y_0 = y]. \]

When \( S \) is a strict local martingale, \( \delta > 0 \) for \( T > 0 \) and \( \delta \) is a solution to (BS-PDE).
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When \( S \) is a strict local martingale, \( \delta > 0 \) for \( T > 0 \) and \( \delta \) is a solution to (BS-PDE). However, \( 0 \) is another solution.

If \( u \) is a solution to (BS-PDE), \( u + \delta \) is another solution.
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When \(S\) is a strict local martingale, \(\delta > 0\) for \(T > 0\) and \(\delta\) is a solution to (BS-PDE). However, 0 is another solution.

If \(u\) is a solution to (BS-PDE), \(u + \delta\) is another solution.

Remark: For a strict local martingale \(S\), coefficients may grow faster than linearly. This is outside the standard framework of classical and viscosity solutions.
Martingale property of $S$

$$d\tilde{Y}_t = \left(\mu(\tilde{Y}_t) + \rho b \sigma(\tilde{Y}_t)\right) dt + \sigma(\tilde{Y}_t) d\tilde{W}_t, \quad Y_0 = y,$$

it has a unique strong solution up to $\zeta^Y = \inf\{t \geq 0 : \tilde{Y}_t^Y = \infty\}$.

Consider $v(x) := \int_c^x \frac{s(x) - s(y)}{s'(y)\sigma^2(y)} dy$, where $s$ is the scale fun. of $\tilde{Y}$.
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Consider \( v(x) := \int_c^x \frac{s(x) - s(y)}{s'(y)\sigma^2(y)} dy \), where \( s \) is the scale fun. of \( \tilde{Y} \).

**Proposition (Sin 98)**

\[ \mathbb{E}[S_T] = S_0 \cdot \mathbb{Q}(\zeta^y > T). \ T.F.A.E. \]

- \( S^{x,y} \) is a martingale for any \((x, y) \in \mathbb{R}^2_{++}\).
- \( v(\infty) = \infty. \)
Martingale property of $S$

$$d\tilde{Y}_t = \left(\mu(\tilde{Y}_t) + \rho b\sigma(\tilde{Y}_t)\right) dt + \sigma(\tilde{Y}_t)d\tilde{W}_t, \quad Y_0 = y,$$

it has a unique strong solution up to $\zeta^\gamma = \inf\{t \geq 0 : \tilde{Y}^\gamma_t = \infty\}$.

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$\triangleright$ $S^{x,y}$ is a martingale for any $(x, y) \in \mathbb{R}^{++}$.

$\triangleright$ $v(\infty) = \infty$.

Proposition

$\triangleright$ $S^x_{\cdot \wedge T}$ is a strict local mart. for some, then all, $(x, y, T) \in \mathbb{R}^{3+}$.

$\triangleright$ $v(\infty) < \infty$.

Remark: This generalizes Theorem 2.4 in [Lions-Musiela 07].
Boundary conditions

Let $h$ be the smallest nonnegative, concave, and nondecreasing function that dominates $g$.

$h$ is of linear or strict sublinear growth whenever $g$ does.

$$u(x, y, T) \leq h(x), \quad (x, y, T) \in \mathbb{R}^3.$$
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Boundary conditions at $y = 0$:

Consider $U^{x,y}_t := u(S_t^x, Y_t^y, T - t) = \mathbb{E} \left[ g(S_T^x) | \mathcal{F}_t \right]$.

It is a martingale on $[0, T]$. 
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Boundary conditions at $y = 0$:

Consider $U_t^{x,y,T} := u(S_t^{x,y}, Y_t^y, T - t) = \mathbb{E} \left[ g(S_T^{x,y}) \mid \mathcal{F}_t \right].$

It is a martingale on $[0, T]$.

For any solution $\nu$, $V_t^{x,y,T} := \nu(S_t^{x,y}, Y_t^y, T - t)$ needs to be at least a local martingale on $[0, T]$. 
Boundary conditions

Let \( h \) be the smallest nonnegative, concave, and nondecreasing function that dominates \( g \).

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Boundary conditions at \( y = 0 \):

Consider \( U_t^{x, y, T} := u(S_t^{x, y}, Y_t^y, T - t) = \mathbb{E}[g(S_T^{x, y}) | \mathcal{F}_t] \).

It is a martingale on \([0, T]\).

For any solution \( v \), \( V_t^{x, y, T} := v(S_t^{x, y}, Y_t^y, T - t) \) needs to be at least a local martingale on \([0, T]\).

If \( v \in C^{2,2,1}(\mathbb{R}_+^3) \), \( V_t^{x, y, T} \) is a local martingale until

\[
  \tau_0^y = \inf \{ t \geq 0 | Y_t^y = 0 \}.
\]

Boundary cond. at \( y = 0 \) \( \implies V_t^{x, y, T} \) is a local mart. until \( T \).
Boundary condition cond.

(A) $\mathcal{Q}[\tau_0^y = \infty] = 1$: no boundary cond. is needed.

(B) $\mathcal{Q}[\tau_0^y < \infty] > 0$ and $\mu(0) = 0$: $v(x, 0, T) = g(x)$.

(C) $\mathcal{Q}[\tau_0^y < \infty] > 0$ and $\mu(0) > 0$: consider the class $\mathcal{C}$. 

A function $v \in \mathcal{C}$ if

(i) $v \in \mathcal{C}(\mathbb{R}^3_+ \cap \mathcal{C}_2^2(\mathbb{R}^3_+) \cap \mathcal{C}_0^1(\mathbb{R}^++\times\mathbb{R}_+))$,

(ii) $\limsup_{y \downarrow 0} b_2(\mathbb{R}^+ v(x, y, T) \mathcal{C})^2 < \infty$ for $(x, T) \in \mathbb{R}^3_+$,

(iii) $0 \leq v(x, y, T) \leq h(x)$ for $(x, y, T) \in \mathbb{R}^3_+$ and

(iv) $\partial_\mathcal{T} v(x, y, T) = L v(x, y, T)$ for $(x, y, T) \in \mathbb{R}^3_+$. 

Let $\mathcal{C}$ is the closure of $\mathcal{C}$ under the point-wise convergence.
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A function $v \in \mathcal{C}$ if

(i) $v \in C(\mathbb{R}_+^3) \cap C^{2,2,1}(\mathbb{R}_+^3) \cap C^{0,1,1}(\mathbb{R}_+^2 \times \mathbb{R}_+ \times \mathbb{R}_+^2)$,

(ii) $\limsup_{y \downarrow 0} b^2(y) \left| \partial_{xx}^2 v(x, y, T) \right| < \infty$ for $(x, T) \in \mathbb{R}_+^2$,

(iii) $0 \leq v(x, y, T) \leq h(x)$ for $(x, y, T) \in \mathbb{R}_+^3$ and

(iv) $\partial_T v(x, y, T) = \mathcal{L} v(x, y, T)$ for $(x, y, T) \in \mathbb{R}_+^3$.

Let $\overline{\mathcal{C}}$ is the closure of $\mathcal{C}$ under the point-wise convergence.
Classical solutions

Definition

$v$ is called a classical solution (with growth domination $h$):

(A) : $v \in C(\mathbb{R}^3_+) \cap C^{2,2,1}(\mathbb{R}^3_{++}), 0 \leq v \leq h$, and $v$ solves (BS-PDE).

(B) : All conditions in (A), $v$ satisfies $v(x, 0, T) = g(x)$.

(C) : $v \in \overline{C} \cap C(\mathbb{R}^3_+)$ and $v$ satisfies the initial cond.
Classical solutions

Definition

\( \nu \) is called a classical solution (with growth domination \( h \)):

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(B) : All conditions in (A), \( \nu \) satisfies \( \nu(x, 0, T) = g(x) \).

(C) : \( \nu \in \overline{C} \cap C(\mathbb{R}_+^3) \) and \( \nu \) satisfies the initial cond.

Remark

- Since \( 0 \leq u \leq h \), it suffices to consider nonnegative solutions dominated by \( h \).

- In Case (C), Schauder interior estimate implies that in fact \( \nu \in C^{2,2,1}(\mathbb{R}_+^3) \) satisfies \( \partial_T \nu = \mathcal{L} \nu \) on \( \mathbb{R}_+^3 \).

- Boundary conditions are specified to identify \( u \) as a unique solution. This is contrast to [Ekstrom-Tysk 10].
Main results

Theorem (Existence)

*The value function $u$ is a classical solution to (BS-PDE). Moreover, it is the smallest classical solution.*
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Theorem (Existence)

*The value function* $u$ *is a classical solution to (BS-PDE).* *Moreover, it is the smallest classical solution.*

Theorem (Uniqueness)

*The following two statements hold:*

(i) *When* $g$ *is of strictly sublinear growth,*

$u$ *is the unique classical soln dominated by* $h$.

(ii) *When* $g$ *is of linear growth,* *T.F.A.E.*

$u$ *is the unique classical soln dominated by* $h$

$S$ *is a martingale.

$v(\infty) = \infty$.

*Uniqueness holds* $\Leftrightarrow$ *the following comparison result holds.*

Let $v$ and $w$ be classical super/sub-solutions.

$v(x, y, 0) \geq g(x) \geq w(x, y, 0) \implies v \geq w$ on $\mathbb{R}^3_+$. 
Remarks

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3. The comparison result is also a necessary condition for uniqueness.
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3. The comparison result is also a necessary condition for uniqueness.

4. In the special case $g(x) \equiv x$, consider $I(y, T) = \mathbb{E}[\mathcal{E}(\int b(Y_s) \, dW_s)_T]$. It satisfies

   \[
   \partial_T I - \frac{1}{2} \sigma^2(y) \partial_{yy} I - (\mu + \rho b \sigma)(y) \partial_y I = 0, \quad I(y, 0) = 1.
   \]

T.F.A.E:

- $I$ is the unique solution among bounded functions.
- $\mathcal{E}(\int b(Y_s) \, dW_s)$ is a martingale.
- $\tilde{Y}$ does not explode to infinity.

Therefore the infinity is natural boundary for $\tilde{Y}$, uniqueness holds without any boundary cond. at infinity.
Proof: Martingale $\iff$ uniqueness (verification)

Lemma

If $v$ is a classical solution, then $V^{x,y,T} = v(S^{x,y}, Y^y, T - \cdot)$ is a local martingale on $[0, T]$.
Proof: Martingale $\implies$ uniqueness (verification)

Lemma
If $\nu$ is a classical solution, then $V^{x,y,T} = \nu(S^{x,y}, Y^y, T - \cdot)$ is a local martingale on $[0, T]$.

When 0 is instantaneous reflect boundary, this follows from:
1. the local time of $Y$ at 0 is 0;
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To show the uniqueness, we want to show $\nu \equiv u$.

Let $\{\sigma^n\}_{n \in \mathbb{N}}$ be a localizing sequence of $V$. 
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Let $\{\sigma^n\}_{n \in \mathbb{N}}$ be a localizing sequence of $V$.

$V_{\sigma^n \wedge T}^{x,y,T} \leq C(1 + S_{\sigma^n \wedge T})$. 
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$$V^{x,y,T}_{\sigma^n \wedge T} \leq C(1 + S_{\sigma^n \wedge T}).$$

$S$ is a martingale $\implies \{S_{\sigma^n \wedge T}\}_{n \in \mathbb{N}}$ is uniformly integrable.
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$$V_{\sigma^n \wedge T}^{x,y,T} \leq C(1 + S_{\sigma^n \wedge T}).$$

$S$ is a martingale $\implies \{S_{\sigma^n \wedge T}\}_{n \in \mathbb{N}}$ is uniformly integrable.

Therefore, we can exchange limit and expectation in

$$v(x, y, T) = \lim_n \mathbb{E} \left[ V_{\sigma^n \wedge T}^{x,y,T} \right] = \mathbb{E}[g(S_T^{x,y})] = u(x, y, T).$$

$\square$
Thanks for your attention!