

## ON THE DYBVIG-INGERSOLL-ROSS THEOREM

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The Dybvig-Ingersoll-Ross (DIR) theorem states that, in arbitrage-free term structure models, long-term yields and forward rates can never fall. We present a refined version of the DIR theorem, where we identify the reciprocal of the maturity date as the maximal order that long-term rates at earlier dates can dominate long-term rates at later dates. The viability assumption imposed on the market model is weaker than those appearing previously in the literature.

KEY WORDS: long maturities, forward rates, Dybvig-Ingersoll-Ross.

### 1. INTRODUCTION

#### 1.1. Background and Discussion of the Results

In interest-rate modeling, it is a well-known result that if the market is arbitrage-free, then long-maturity yields, as well as forward rates, can never fall. The last statement is commonly referred to as the Dybvig-Ingersoll-Ross (DIR) theorem, acknowledging the fact that its first occurrence was in Dybvig, Ingersoll, and Ross (1996) and opened this research direction. Since then, there has been substantial interest in the literature regarding this result: McCulloch (2000) contained some clarifications on the original proof. Later, Hubalek, Klein, and Teichmann (2002) presented an elegant mathematical proof in a quite general context. Recently, Goldammer and Schmock (2010) discussed further interesting generalizations, as well as an asymptotic minimality property, also appearing in Schulze (2008).

In order to get a better feeling for what the DIR theorem states, let  $P_t^T$  denote the price at time  $t \in \mathbb{R}_+$  of a zero-coupon bond with maturity  $T > t$ ; then,

$$(1.1) \quad R_t^T = -\frac{\log(P_t^T)}{T-t}$$

is the prevailing yield from time  $t$  to maturity  $T$ . By “long-maturity yield at time  $t$ ,” one usually means the limit of  $R_t^T$  as  $T \rightarrow \infty$ , which, provided it can be defined in some sense, we denote by  $R_t^\infty$ . The DIR theorem states that, under the assumption

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of absence of arbitrages in the market,  $R_s^\infty \leq R_t^\infty$  holds whenever  $s \leq t$ . A completely similar statement is valid for forward rates; to refrain ourselves from being repetitive, we shall focus on yields for the purposes of the introductory discussion here.

Originally, the DIR theorem is stated for term-structure models of interest rates. We choose here to take the more comprehensive viewpoint of the term-structure of a market for exchange over time of some underlying asset, which could be a currency, a commodity with investment value, or a similar security. Within this framework,  $P_t^T$  represents the units of the underlying asset required by the market at time  $t \in \mathbb{R}_+$  in return of one unit of the underlying asset at time  $T > t$ . In other words,  $P_t^T$  denotes the price, in units of the asset, of a derivative contract that allows transferring the asset through time; as such, it is therefore deeply linked to the term structure of yields and forward rates.

Having clarified the background and statement of the DIR theorem in this general context, two natural questions come to mind.

1. What can we salvage if  $R_t^\infty$  cannot be defined for some  $t \in \mathbb{R}_+$ , that is, if limits of yields as the maturity tends to infinity do not exist?
2. For long-term, but finite maturities  $T$ , the relation  $R_s^T \leq R_t^T$ , for  $s \leq t$ , might fail to hold. How large can the discrepancy  $R_s^T - R_t^T$  be?

An approach to answering the first question is undertaken in Goldammer and Schmock (2010). There, an appropriate superior limit definition is utilized in order to compensate for the possible nonexistence of the actual limit. In fact, the authors give a reasonable economic justification for considering the aforementioned superior limit. The approach we take here is to consider the difference  $R_s^T - R_t^T$  for  $s \leq t$  as  $T \rightarrow \infty$ , and examine when its superior limit (in probability) exists and is nonpositive. Though the previous two approaches are similar in nature, focusing on the difference of the rates allows for more detailed comparisons. An example of such instance would be the case where long-term rates explode in the limit.

To the best of our knowledge, an attempt to answer the second question posed above has not appeared in the literature. We show here that the highest possible order that  $R_s^T$  can be larger than  $R_t^T$  is  $1/T$ , that is, the reciprocal of the long-term maturity. In fact, we shall show by example that this order is the best possible that can be achieved.

As mentioned earlier, and as easy counterexamples show, the DIR theorem is valid only under an assumption regarding nonexistence of some sort of arbitrages in the market. In the literature, there had been mainly two approaches in formalizing such an assumption.

- In the first approach, authors stipulate a “no limiting arbitrage” condition in the market, reminiscent of the “No Free Lunch with Vanishing Risk” condition introduced in Delbaen and Schachermayer (1994). This was for example the approach initially taken in Dybvig, Ingersoll, and Ross (1996), as well as in McCulloch (2000) shortly after. More recently, Schulze (2008) also takes the same path.
- The second approach is to assume the existence of a locally equivalent martingale measure (EMM) in the market. Bond prices are defined as expectations under the EMM of contingent claims giving unit payoff at maturity, discounted by the savings account. This viewpoint on the statement of the DIR theorem was initiated in Hubalek, Klein, and Teichmann (2002).

The Fundamental Theorem of Asset Pricing, established in Delbaen and Schachermayer (1994) for the case of equity markets, indicates that the previous assumptions are very closely connected. However, the fact that a continuum of assets is available to trade in

bond markets forces different tools to be employed under the two approaches above. This is true even in papers who treat both cases, like in Goldammer and Schmock (2010).

Here, we take a path that unifies the above two approaches, at the same time weakening market viability assumptions that have previously appeared. This is done by assuming existence of “strictly positive supermartingale deflators” in the market, an assumption weaker than the existence of an EMM, and equivalent to absence of arbitrages of the first kind in the bond market where only long positions are allowed, as is discussed in Kardaras (2011).

After a few probabilistic definitions and later needed results will conclude this section, the structure of the remaining paper is as follows: In Section 2 all the results are presented, while Section 3 contains examples that illustrate our main findings.

### 1.2. Probabilistic Definitions and Notation

Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space where all the random elements appearing below will be based. All relationships between random variables (like equalities, inequalities, etc.) should be understood to hold in the  $\mathbb{P}$ -a.s. sense.

For  $A \in \mathcal{F}$  and  $B \in \mathcal{F}$ , we write  $A \subseteq_{\mathbb{P}} B$  if and only if  $\mathbb{P}[(\Omega \setminus B) \cap A] = 0$ —in other words,  $A \subseteq_{\mathbb{P}} B$  means that  $A$  is contained in  $B$  modulo  $\mathbb{P}$ . Also,  $A =_{\mathbb{P}} B$  means  $A$  and  $B$  are equal modulo  $\mathbb{P}$ , that is, that both  $A \subseteq_{\mathbb{P}} B$  and  $B \subseteq_{\mathbb{P}} A$  hold.

For a collection  $(\xi^T)_{T \in \mathbb{R}_+}$  of random variables,  $\mathbb{P}\text{-lim sup}_{T \rightarrow \infty} \xi^T$  is defined to be the essential infimum of all extended-valued random variables (i.e., random variables that can potentially take infinite values, both positive and negative)  $\zeta$  such that  $\lim_{T \rightarrow \infty} \mathbb{P}[\xi^T \leq \zeta] = 1$ . Observe that  $\mathbb{P}\text{-lim sup}_{T \rightarrow \infty} \xi^T$  is an extended-valued random variable. We also define  $\mathbb{P}\text{-lim inf}_{T \rightarrow \infty} \xi^T := -\mathbb{P}\text{-lim sup}_{T \rightarrow \infty} (-\xi^T)$ . The limit in probability of  $(\xi^T)_{T \in \mathbb{R}_+}$  as  $T \rightarrow \infty$  exists if and only if  $\mathbb{P}\text{-lim inf}_{T \rightarrow \infty} \xi^T = \mathbb{P}\text{-lim sup}_{T \rightarrow \infty} \xi^T$ ; in this case, this limit is denoted by  $\mathbb{P}\text{-lim}_{T \rightarrow \infty} \xi^T$ . For these definitions and more discussion, we refer the reader to He, Wang, and Yan (1992, chapter I).

Let again  $(\xi^T)_{T \in \mathbb{R}_+}$  be a collection of random variables. Whenever

$$\lim_{\ell \rightarrow \infty} \left( \limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T > \ell] \right) = 0$$

holds, we shall be writing  $\xi^T = O_{\mathbb{P}}^{\uparrow}(1)$  as  $T \rightarrow \infty$ . Also, if  $(\alpha^T)_{T \in \mathbb{R}_+}$  is a sequence of strictly positive real numbers and  $A \in \mathcal{F}$ , we write  $\xi^T = O_{\mathbb{P}}^{\uparrow}(\alpha^T)$  on  $A$  as  $T \rightarrow \infty$  if and only if  $\mathbb{I}_A \xi^T / \alpha^T = O_{\mathbb{P}}^{\uparrow}(1)$  as  $T \rightarrow \infty$ , where  $\mathbb{I}_A$  denotes the indicator function of the event  $A$ . Furthermore, we write  $\xi^T = O_{\mathbb{P}}^{\downarrow}(\alpha^T)$  on  $A$  as  $T \rightarrow \infty$  if and only if  $-\xi^T = O_{\mathbb{P}}^{\uparrow}(\alpha^T)$  on  $A$  as  $T \rightarrow \infty$ . Finally,  $\xi^T = O_{\mathbb{P}}(\alpha^T)$  on  $A$  as  $T \rightarrow \infty$  means  $|\xi^T| = O_{\mathbb{P}}^{\uparrow}(\alpha^T)$  on  $A$  as  $T \rightarrow \infty$ . If the set  $A \in \mathcal{F}$  is not explicitly mentioned, it will be tacitly assumed that  $A = \Omega$ .

As the reader might have already guessed, we are using throughout the upwards-pointing arrow “ $\uparrow$ ” as a mnemonic device when dealing with boundedness from above; similarly, the downwards-pointing arrow “ $\downarrow$ ” will be used in cases where boundedness from below is involved.

We close this introductory discussion with two general results, which will be used in the text.

**PROPOSITION 1.1.** *In the statements below,  $(\xi^T)_{T \in \mathbb{R}_+}$  and  $(\zeta^T)_{T \in \mathbb{R}_+}$  are collections of random variables and  $(\alpha^T)_{T \in \mathbb{R}_+}$  is a collection of strictly positive real numbers.*

- (1)  $\mathbb{P}\text{-lim sup}_{T \rightarrow \infty} \xi^T < \infty$  implies that  $\xi^T = O_{\mathbb{P}}^{\uparrow}(1)$  as  $T \rightarrow \infty$ .

- (2) If  $\xi^T = O_{\mathbb{P}}^{\uparrow}(\alpha^T)$  as  $T \rightarrow \infty$  and  $\lim_{T \rightarrow \infty} \alpha^T = 0$ , then  $\mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \xi^T \leq 0$ .
- (3) If  $\mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} (\xi^T - \zeta^T) \leq 0$ , then  $\mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \xi^T \leq \mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \zeta^T$ .

*Proof.*

- (1) Let  $\bar{\xi} = \mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \xi^T$ . Fix  $\ell \in \mathbb{R}_+$ . The set-inclusion  $\{\bar{\xi} \leq \ell - 1\} \cap \{\xi^T \leq \bar{\xi} + 1\} \subseteq \{\xi^T \leq \ell\}$ , valid for all  $T \in \mathbb{R}_+$ , gives

$$(1.2) \quad \mathbb{P}[\xi^T > \ell] \leq \mathbb{P}[\bar{\xi} > \ell - 1] + \mathbb{P}[\xi^T > \bar{\xi} + 1].$$

As  $\mathbb{P}[\bar{\xi} < \infty] = 1$ , we get  $\lim_{T \rightarrow \infty} \mathbb{P}[\xi^T > \bar{\xi} + 1] = 0$  from the definition of  $\mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \xi^T$ . Therefore, (1.2) gives  $\limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T > \ell] \leq \mathbb{P}[\bar{\xi} > \ell - 1]$ . Using again  $\mathbb{P}[\bar{\xi} < \infty] = 1$  we get  $\lim_{\ell \rightarrow \infty} \mathbb{P}[\bar{\xi} > \ell - 1] = 0$ ; therefore,  $\lim_{\ell \rightarrow \infty} (\limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T > \ell]) = 0$ , which is what we needed to prove.

- (2) Let  $\epsilon > 0$ . Then,

$$\limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T > \epsilon] = \limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T / \alpha^T > \epsilon / \alpha^T] \leq \limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T / \alpha^T > \ell]$$

holds for all  $\ell > 0$  in view of  $\lim_{T \rightarrow \infty} \alpha^T = 0$ . Taking limits as  $\ell \rightarrow \infty$  in the extreme sides of the previous inequality we obtain  $\limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T > \epsilon] = 0$ , which means that  $\mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \xi^T \leq \epsilon$ . As this holds for all  $\epsilon > 0$ , we get  $\mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \xi^T \leq 0$ .

- (3) Take any random variable  $\eta$  such that  $\lim_{T \rightarrow \infty} [\zeta^T \leq \eta] = 1$ . For any  $\epsilon > 0$ , we have

$$\limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T > \epsilon + \eta] \leq \limsup_{T \rightarrow \infty} \mathbb{P}[\zeta^T > \eta] + \limsup_{T \rightarrow \infty} \mathbb{P}[\xi^T - \zeta^T > \epsilon] = 0.$$

This implies that  $\mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \xi^T \leq \epsilon + \mathbb{P}\text{-}\lim \sup_{T \rightarrow \infty} \zeta^T$  for all  $\epsilon > 0$ . Letting now  $\epsilon$  tend to zero, we get the result. □

**PROPOSITION 1.2.** *Let  $(\xi^T)_{T \in \mathbb{R}_+}$  be a collection of random variables. Then, the following statements are true.*

- (1) There exists  $\Phi^{\downarrow} \in \mathcal{F}$  such that:  $\xi^T = O_{\mathbb{P}}^{\downarrow}(1)$  on  $A \in \mathcal{F}$  as  $T \rightarrow \infty$  if and only if  $A \subseteq_{\mathbb{P}} \Phi^{\downarrow}$ .
- (2) There exists  $\Phi^{\uparrow} \in \mathcal{F}$  such that:  $\xi^T = O_{\mathbb{P}}^{\uparrow}(1)$  on  $A \in \mathcal{F}$  as  $T \rightarrow \infty$  if and only if  $A \subseteq_{\mathbb{P}} \Phi^{\uparrow}$ .
- (3) There exists  $\Phi \in \mathcal{F}$  such that:  $\xi^T = O_{\mathbb{P}}(1)$  on  $A \in \mathcal{F}$  as  $T \rightarrow \infty$  if and only if  $A \subseteq_{\mathbb{P}} \Phi$ .

Furthermore, the sets  $\Phi^{\downarrow}$ ,  $\Phi^{\uparrow}$ , and  $\Phi$  are unique modulo  $\mathbb{P}$ .

*Proof.* We only prove statement (1); the proofs of statement (2) and statement (3) are entirely similar.

Consider the class  $\mathcal{G}^{\downarrow} := \{A \in \mathcal{F} \mid \xi^T = O_{\mathbb{P}}^{\downarrow}(1) \text{ holds on } A \text{ as } T \rightarrow \infty\} \subseteq \mathcal{F}$ . Since  $\emptyset \in \mathcal{G}^{\downarrow}$ , the class  $\mathcal{G}^{\downarrow}$  is nonempty. Furthermore, it is relatively straightforward to see that  $\mathcal{G}^{\downarrow}$  is closed under countable unions. Observe that  $\subseteq_{\mathbb{P}}$  is a partial ordering on the subsets of  $\mathcal{F}$ . Let  $\mathcal{H} \subseteq \mathcal{G}^{\downarrow}$  be a totally ordered set for the order  $\subseteq_{\mathbb{P}}$  and let  $p := \sup\{\mathbb{P}[A] \mid A \in \mathcal{H}\}$ . For all  $n \in \mathbb{N}$ , pick  $A^n \in \mathcal{H}$  such that  $\mathbb{P}[A^n] \geq p - 1/n$ . If  $A := \bigcup_{n \in \mathbb{N}} A^n$ , then  $A \in \mathcal{G}^{\downarrow}$  and it is straightforward that  $A$  is an upper bound of  $\mathcal{H}$ . Zorn’s lemma then implies the existence of a maximal element in  $\mathcal{G}^{\downarrow}$ . Since  $\mathcal{G}^{\downarrow}$  is closed with respect to finite unions, we conclude that the previous maximal element is unique, which we call  $\Phi^{\downarrow}$ . The uniqueness modulo  $\mathbb{P}$  of such set  $\Phi^{\downarrow}$  follows immediately from statement (1) of the result. □

## 2. RESULTS

### 2.1. Market Model and Yields

In all that follows, we shall be working on a filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ , where  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$  is a right-continuous filtration. We consider a collection  $(P^T)_{T \in \mathbb{R}_+}$  of càdlàg (right continuous with left-hand limits) adapted stochastic processes indexed by their maturity  $T \in \mathbb{R}_+$ . For each  $T \in \mathbb{R}_+$ ,  $P^T$  is defined in the finite time interval  $[0, T]$ , that is,  $P^T = (P_t^T)_{t \in [0, T]}$ . We assume that  $\mathbb{P}[P_t^T > 0] = 1$  holds for all  $t \in [0, T]$  and  $T \in \mathbb{R}_+$ , as well as  $\mathbb{P}[P_T^T = 1] = 1$ . For a concrete interpretation, regard  $P_t^T$  as the price at time  $t$  of an instrument delivering a unit of account at time  $T \geq t$ . Observe however that we do not necessarily assume that  $P^T \leq 1$ , which is true in bond markets. This is done for a number of reasons:

- (1) From a theoretical viewpoint,  $P^T \leq 1$  is not needed for the results we shall present.
- (2) From a model-building perspective, such assumption would immediately disqualify all Gaussian short-rate models that are widely used in the industry.
- (3) On a more practical side, and as mentioned in the Introduction, our results are applicable in diverse situations, such as commodity markets. If the storage costs that apply for the commodity involved, which could be for example oil, are more than the convenience yield it carries, it is certainly possible that  $P_t^T > 1$  holds for  $t < T$ .

For  $0 \leq t < T$ , the yield  $R_t^T$  from time  $t$  to maturity  $T$  is defined in (1.1). Events where long-term yields are essentially bounded will turn out to be crucial in our discussion. In all that follows, for  $t \in \mathbb{R}_+$ , we use  $\Phi_t^\downarrow$ ,  $\Phi_t^\uparrow$ , and  $\Phi_t$  to be the events appearing in the statement of Proposition 1.2 corresponding to the case where  $\xi^T = R_t^T$  for  $T > t$ . It is apparent that  $\Phi_t^\downarrow$  is the maximal (modulo  $\mathbb{P}$ ) event such that long-term yields at time  $t$  are bounded in probability from below. Exactly similar characterizations are true for  $\Phi_t^\uparrow$  and  $\Phi_t$  in terms of boundedness in probability from above and two-sided boundedness in probability, respectively. Obviously,  $\Phi_t =_{\mathbb{P}} \Phi_t^\downarrow \cap \Phi_t^\uparrow$  holds for all  $t \in \mathbb{R}_+$ .

**REMARK 2.1.** In bond markets, we have  $P^T \leq 1$  for all  $T \in \mathbb{R}_+$ , or equivalently that  $R^T \geq 0$  for all  $T \in \mathbb{R}_+$ . Therefore,  $\Phi_t^\downarrow =_{\mathbb{P}} \Omega$  for all  $t \in \mathbb{R}_+$ ; in other words, long-term yields trivially are essentially bounded from below at every time  $t \in \mathbb{R}_+$ .

**REMARK 2.2.** It has been empirically observed that yield curves flatten out for very long maturities; a discussion on this appears for example in Malkiel (1966). There also exist theoretical justifications of this phenomenon, as is described in El Karoui, Frachot, and Geman (1998) and Yao (1999). To rigorously describe such behavior in a weak sense, assume that  $\mathbb{P}\text{-}\lim_{T \rightarrow \infty} R_t^T$  exists and is a  $\mathbb{P}$ -a.s. finite random variable for a fixed  $t \in \mathbb{R}_+$ . Then, statement (1) of Proposition 1.1 implies that  $\Phi_t =_{\mathbb{P}} \Omega$ .

**REMARK 2.3.** In this paper, we treat continuous-time models—for this reason, we use the definition (1.1) for yields. We note, however, that all our results still hold in discrete-time (infinite horizon) settings, with the appropriate changes in the definition of yields and forward rates (see, e.g., equations (2.1) and (2.3) of Goldammer and Schmock 2010). The details have been extensively discussed in Hubalek, Klein, and Teichmann (2002) and Goldammer and Schmock (2010), where we refer the interested reader.

### 2.2. Strictly Positive Supermartingale Deflators

The notion introduced below is central in our discussion.

**DEFINITION 2.4.** *A strictly positive supermartingale deflator in the market is a càdlàg process  $Y$  with  $\inf_{t \in [0, T]} Y_t > 0$ ,  $\mathbb{P}$ -a.s., for all  $T \in \mathbb{R}_+$ , such that  $(Y_t P_t^T)_{t \in [0, T]}$  is a supermartingale for all  $T \in \mathbb{R}_+$ .*

Existence of a strictly positive supermartingale deflator is equivalent to absence of arbitrages of the first kind in the market with acting investors that may only take long positions on the instruments with prices  $(P^T)_{T \in \mathbb{R}_+}$ . For such “abstract” markets with infinite number of assets, the last fact is explained in detail in Kardaras (2011).

**REMARK 2.5.** Even if the processes  $(P^T)_{t \in [0, T]}$  for  $T \in \mathbb{R}_+$  are not initially assumed to have càdlàg paths, but are only right-continuous in probability, the existence of a strictly positive supermartingale deflator, as in Definition 2.4, coupled with the standard supermartingale modification theorem (which can be applied here because the underlying filtration is right-continuous), implies that there exist càdlàg modifications of  $(P^T)_{t \in [0, T]}$ ,  $T \in \mathbb{R}_+$ . As every model encountered in practice consists of càdlàg price-processes, we plainly enforce this requirement from the outset.

We shall now discuss the traditional way of constructing markets possessing a strictly positive supermartingale deflator, via the existence of an EMM. We include this discussion for completeness since we shall be using it in the examples below. It is important to note that markets where a strictly positive supermartingale deflator exists form a wide-encompassing class, substantially larger than the concrete situation described in the example below. A concrete realistic example where an EMM fails to exist, but a strictly positive supermartingale deflator does exist, is presented in subsection 3.2 of Bruti-Liberati, Nikitopoulos-Sklibosios, and Platen (2010); in this respect, see also Subsection 3.3 of this paper.

**EXAMPLE 2.6.** *Let  $\mathbb{Q}$  be a probability on  $(\Omega, \mathcal{F})$  such that  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  on  $\mathcal{F}_t$  for all  $t \in \mathbb{R}_+$ . Consider also a càdlàg nonnegative process  $B$ , representing the savings account, such that  $\mathbb{P}[\inf_{t \in [0, T]} B_t > 0] = 1$  as well as  $\mathbb{E}^{\mathbb{Q}}[1/B_T] < \infty$ , for all  $T \in \mathbb{R}_+$ . Define  $P^T$  to be a càdlàg modification (which exists in view of the fact that the underlying filtration is right-continuous) of the process  $[0, T] \ni t \mapsto B_t \mathbb{E}^{\mathbb{Q}}[1/B_T | \mathcal{F}_t]$ . For this market, a strictly positive supermartingale deflator exists and is given by*

$$Y := \frac{1}{B} \frac{d(\mathbb{Q} |_{\mathcal{F}})}{d(\mathbb{P} |_{\mathcal{F}})}.$$

*(In fact, one should consider a càdlàg version of the process above.) Indeed, it is straightforward to check that  $(Y_t P_t^T)_{t \in [0, T]}$  is actually a  $\mathbb{P}$ -martingale for all  $T \in \mathbb{R}_+$ .*

Contrary to the construction in Example 2.6 above, we do not explicitly define a savings account here, as it is not needed. At any rate, given a market with prices  $(P^T)_{T \in \mathbb{R}_+}$ , if a savings account  $B$  is able to generate the market in the sense of Example 2.6, that is, if  $P_t^T = B_t \mathbb{E}^{\mathbb{Q}}[1/B_T | \mathcal{F}_t]$  holds for all  $t \leq T$  where  $\mathbb{Q}$  is equivalent to  $\mathbb{P}$  on  $\mathcal{F}_t$  for all  $t \in \mathbb{R}_+$ , then  $B$  is essentially unique; see Döberlein, Schweizer, and Stricker (2000).

### 2.3. Long-term Yields

We are ready to state the main result of the paper, which can be regarded as a ramification of the DIR theorem.

**THEOREM 2.7.** *Suppose that a strictly positive supermartingale deflator exists in the market. Let  $s \leq t$ . Then:*

- (1)  $\Phi_s^\downarrow \subseteq_{\mathbb{P}} \Phi_t^\downarrow$ .
- (2)  $R_s^T - R_t^T = O_{\mathbb{P}}^\uparrow(1/T)$  holds on  $\Phi_t^\downarrow$  as  $T \rightarrow \infty$ .

*Proof.* For all  $T \in \mathbb{R}_+$ , define  $L^T = (L_t^T)_{t \in [0, T]}$  via  $L^T := YP^T$ , where  $Y$  is a strictly positive supermartingale deflator as in Definition 2.4. Then,  $(L_t^T)_{t \in [0, T]}$  is a nonnegative supermartingale and  $(T - u)R_u^T = -\log(L_u^T) + \log(Y_u)$  holds whenever  $u < T$ . Write

$$(2.1) \quad (T - t)(R_s^T - R_t^T) = -(t - s)R_s^T + \log\left(\frac{L_t^T}{L_s^T}\right) - \log\left(\frac{Y_t}{Y_s}\right).$$

Let  $\ell > 0$ ; then, we have

$$\mathbb{P}[\log(L_t^T/L_s^T) > \ell] = \mathbb{P}[L_t^T/L_s^T > e^\ell] \leq e^{-\ell},$$

following from Markov’s inequality, since  $L^T$  is a nonnegative supermartingale. This implies that  $\log(L_t^T/L_s^T) = O_{\mathbb{P}}^\uparrow(1)$  as  $T \rightarrow \infty$ . Since  $\log(Y_t/Y_s)$  is an  $\mathbb{R}$ -valued random variable and  $-(t - s)R_s^T = O_{\mathbb{P}}^\uparrow(1)$  holds on  $\Phi_s^\downarrow$  as  $T \rightarrow \infty$ , (2.1) gives that  $(T - t)(R_s^T - R_t^T) = O_{\mathbb{P}}^\uparrow(1)$  on  $\Phi_s^\downarrow$  as  $T \rightarrow \infty$ . As this obviously implies that  $R_s^T - R_t^T = O_{\mathbb{P}}^\uparrow(1)$  on  $\Phi_s^\downarrow$  as  $T \rightarrow \infty$ , we obtain that

$$T(R_s^T - R_t^T) = (T - t)(R_s^T - R_t^T) + t(R_s^T - R_t^T) = O_{\mathbb{P}}^\uparrow(1) \text{ holds on } \Phi_s^\downarrow \text{ as } T \rightarrow \infty,$$

which is the same as saying that  $R_s^T - R_t^T = O_{\mathbb{P}}^\uparrow(1/T)$  on  $\Phi_s^\downarrow$  as  $T \rightarrow \infty$ . This immediately implies that  $\Phi_s^\downarrow \subseteq_{\mathbb{P}} \Phi_t^\downarrow$ .

Up to now we have proved that  $R_s^T - R_t^T = O_{\mathbb{P}}^\uparrow(1/T)$  on  $\Phi_s^\downarrow$  as  $T \rightarrow \infty$ ; we would like to extend the last relationship to hold on  $\Phi_t^\downarrow$ . Provided that we replace (2.1) with

$$(T - s)(R_s^T - R_t^T) = -(t - s)R_t^T + \log\left(\frac{L_t^T}{L_s^T}\right) - \log\left(\frac{Y_t}{Y_s}\right),$$

one can follow essentially the same steps as above to finish the proof. □

### 2.4. The DIR Theorem Revisited

Let  $s \leq t$ . Theorem 2.7 coupled with statement (2) of Proposition 1.1 immediately gives that  $\mathbb{P}\text{-}\limsup_{T \rightarrow \infty}(R_s^T - R_t^T) \leq 0$  holds on  $\Phi_t^\downarrow$ . In particular, and using statement (3) of Proposition 1.1, we obtain that

$$(2.2) \quad \mathbb{P}\text{-}\limsup_{T \rightarrow \infty} R_s^T \leq \mathbb{P}\text{-}\limsup_{T \rightarrow \infty} R_t^T \text{ holds on } \Phi_t^\downarrow.$$

The last equation (2.2) should be compared with the result obtained in Goldammer and Schmock (2010). Of course, in Goldammer and Schmock (2010) the superior limit is taken in a stronger sense and the assumption that we are working on  $\Phi_t^\downarrow$  is not present. It is



indeed true that (2.2) can be still valid outside of  $\Phi_t^\downarrow$ , even though  $\mathbb{P}\text{-lim sup}_{T \rightarrow \infty} (R_s^T - R_t^T) > 0$ . Such a situation is described in Subsection 3.1.2; there, both sides of (2.2) are equal to infinity, and are, therefore, equal in a trivial sense. Theorem 2.7 refines the asymptotic relationship (2.2) by precisely examining the behavior of the relative differences of long-term yields through different points in time.

### 2.5. Forward Rates

The next aim is to obtain an equivalent of Theorem 2.7 for forward rates, which we now introduce. For  $0 < t < t' \leq T$ , the forward rate, set at time  $t$  for investment from time  $t'$  up to maturity  $T$ , is defined via

$$(2.3) \quad F_{t,t'}^T := \frac{1}{T-t'} \log \left( \frac{P_t^{t'}}{P_t^T} \right) = \frac{T-t}{T-t'} R_t^T - \frac{t'-t}{T-t'} R_{t'}^T.$$

Roughly speaking, the next result we shall present states that yields are essentially bounded exactly on the set where forward rates and yields are asymptotically, as  $T \rightarrow \infty$ , equivalent of order  $1/T$ . Similar statements hold for boundedness from below and above. Observe that there is no market viability assumption in the statement of Proposition 2.8.

**PROPOSITION 2.8.** *Let  $t \in \mathbb{R}_+$  and  $A \in \mathcal{F}$ . The following conditions are equivalent*

- (1)  $R_t^T = O_{\mathbb{P}}^\downarrow(1)$  on  $A$  as  $T \rightarrow \infty$ .
- (2) For all  $t' > t$ ,  $F_{t,t'}^T - R_t^T = O_{\mathbb{P}}^\downarrow(1/T)$  holds on  $A$  as  $T \rightarrow \infty$ .
- (3) For some  $t' > t$ ,  $F_{t,t'}^T - R_t^T = O_{\mathbb{P}}^\downarrow(1/T)$  holds on  $A$  as  $T \rightarrow \infty$ .

*The same equivalences hold if we replace “ $O_{\mathbb{P}}^\downarrow$ ” with “ $O_{\mathbb{P}}^\uparrow$ ” in all conditions (1), (2), and (3), and similarly if we replace “ $O_{\mathbb{P}}^\downarrow$ ” with “ $O_{\mathbb{P}}$ ” in all conditions (1), (2), and (3).*

*Proof.* We shall only prove the equivalence of (1), (2), and (3) as explicitly stated in Proposition 2.8. The treatment in the cases where we replace “ $O_{\mathbb{P}}^\uparrow$ ” with “ $O_{\mathbb{P}}^\downarrow$ ” or “ $O_{\mathbb{P}}$ ” in all conditions (1), (2), and (3) is entirely similar. In what follows,  $t \in \mathbb{R}_+$  and  $A \in \mathcal{F}$  are fixed.

Start by assuming (1) and fix  $t' > t$ . First of all, observe that  $F_{t,t'}^T = O_{\mathbb{P}}^\downarrow(1)$  on  $A$  as  $T \rightarrow \infty$ , as follows from the fact that  $R_{t'}^T = O_{\mathbb{P}}^\downarrow(1)$  on  $A$  as  $T \rightarrow \infty$  and the definition of the forward rates at (2.3). Now,  $(T-t')(F_{t,t'}^T - R_t^T) = (t'-t)(R_t^T - R_{t'}^T)$  as follows again from (2.3), immediately gives that  $(T-t')(F_{t,t'}^T - R_t^T) = O_{\mathbb{P}}^\downarrow(1)$  on  $A$  as  $T \rightarrow \infty$ , since  $R_t^T = O_{\mathbb{P}}^\downarrow(1)$  on  $A$  as  $T \rightarrow \infty$ . Using also the fact that  $F_{t,t'}^T - R_t^T = O_{\mathbb{P}}^\downarrow(1)$  on  $A$  as  $T \rightarrow \infty$ , we get that  $T(F_{t,t'}^T - R_t^T) = O_{\mathbb{P}}^\downarrow(1)$  on  $A$  as  $T \rightarrow \infty$ , which is what we needed to show.

Of course, condition (2) implies condition (3).

Now, assume (3). Observe first that

$$(T-t')(F_{t,t'}^T - R_t^T) = \left( \frac{T-t'}{T} \right) T(F_{t,t'}^T - R_t^T) = O_{\mathbb{P}}^\downarrow(1) \text{ holds on } A \text{ as } T \rightarrow \infty.$$

Then,

$$R_t^T = \left( \frac{T-t'}{t'-t} \right) (F_{t,t'}^T - R_t^T) + R_{t'}^T = O_{\mathbb{P}}^\downarrow(1) \text{ holds on } A \text{ as } T \rightarrow \infty,$$

which is exactly condition (1) and concludes the proof. □



According to Proposition 2.8 and Proposition 1.2,  $\Phi_t^\downarrow$  can be regarded as the largest set where  $F_{t,t'}^T - R_t^T = O_{\mathbb{P}}^\downarrow(1/T)$  holds for some, and then for all,  $t' > t$ . Similar interpretations are valid for the events  $\Phi_t^\uparrow$  and  $\Phi_t$ , where  $t \in \mathbb{R}_+$ .

We are now ready to state the version of Theorem 2.7 for forward rates. The situation is only slightly more complicated, since we have to control the boundedness of yields from both sides at different points in time.

**THEOREM 2.9.** *Suppose that a strictly positive supermartingale deflator exists in the market. Let  $s \leq t$ , as well as  $s < s'$  and  $t < t'$ . Then,  $F_{s,s'}^T - F_{t,t'}^T = O_{\mathbb{P}}^\uparrow(1/T)$  holds on  $\Phi_s^\uparrow \cap \Phi_t^\downarrow$  as  $T \rightarrow \infty$ .*

*Proof.* Write

$$F_{s,s'}^T - F_{t,t'}^T = (F_{s,s'}^T - R_s^T) - (F_{t,t'}^T - R_t^T) + (R_s^T - R_t^T).$$

Now,  $F_{s,s'}^T - R_s^T = O_{\mathbb{P}}^\uparrow(1/T)$  and  $R_t - F_{t,t'}^T = O_{\mathbb{P}}^\uparrow(1/T)$  and both hold on  $\Phi_s^\uparrow \cap \Phi_t^\downarrow$  as  $T \rightarrow \infty$  in view of Proposition 2.8. Furthermore,  $R_s^T - R_t^T = O_{\mathbb{P}}^\uparrow(1/T)$  holds on  $\Phi_t^\downarrow$  by Theorem 2.7. Putting everything together, we obtain the claim of Theorem 2.9.  $\square$

**REMARK 2.10.** Let  $s \leq t$ . If a strictly positive supermartingale deflator exists in the market, statement (1) of Theorem 2.7 gives  $\Phi_s =_{\mathbb{P}} \Phi_s^\uparrow \cap \Phi_s^\downarrow \subseteq_{\mathbb{P}} \Phi_s^\uparrow \cap \Phi_t^\downarrow$ . In particular, Theorem 2.9 implies that  $F_{s,s'}^T - F_{t,t'}^T = O_{\mathbb{P}}^\uparrow(1/T)$  holds on  $\Phi_s$  as  $T \rightarrow \infty$ , whenever  $s < s'$  and  $t < t'$ , which is a more pleasant statement.

### 3. REMARKS AND EXAMPLES

We proceed with several remarks and (counter)examples regarding our main results. The most important ones are given in Subsection 3.2, where it is shown that the reciprocal of the maturity is indeed the best order of domination that can be obtained, and Subsection 3.3, where we demonstrate that our market viability assumption is strictly weaker than the ones that previously appeared in the literature.

#### 3.1. Counterexamples on the Main Results

**3.1.1.** The inclusion  $\Phi_s^\downarrow \subseteq_{\mathbb{P}} \Phi_t^\downarrow$  in Theorem 2.7 might fail when a strictly positive supermartingale deflator does not exist. Consider, for example, the deterministic market with  $P_t^T = 1$  for  $0 \leq t < 1$  and  $t \leq T$ , while  $P_t^T = \exp(T^2 - t^2)$  for  $1 \leq t \leq T$ . Then,  $R_0^T = 0$  and  $R_1^T = -T - 1$  holds for  $T \geq 1$ . Therefore,  $\Phi_0^\downarrow =_{\mathbb{P}} \Omega \not\subseteq_{\mathbb{P}} \emptyset =_{\mathbb{P}} \Phi_1^\downarrow$ .

**3.1.2.** Even when a strictly positive supermartingale deflator exists, the asymptotic behavior of yield differences mentioned in statement (2) of Theorem 2.7 can fail to hold outside  $\Phi_t^\downarrow$ . With  $\mathbb{Q} = \mathbb{P}$  and  $B$  defined via  $B_t = \exp(-t^2)$  for  $t \in \mathbb{R}_+$ , define a market according to Example 2.6. In this case,  $\Phi_t^\downarrow =_{\mathbb{P}} \emptyset$  for all  $t \in \mathbb{R}_+$ . Further,  $R_t^T = -T - t$  for  $t \leq T$ , which implies that  $R_s^T - R_t^T = t - s > 0$  for  $s < t$ , and statement (2) of Theorem 2.7 fails to hold. Observe also in this example that the asymptotic relationship  $\limsup_{T \rightarrow \infty} R_s^T = -\infty = \limsup_{T \rightarrow \infty} R_t^T$  trivially holds identically; however, one cannot honestly claim that long-term yields are nonincreasing, as  $\lim_{T \rightarrow \infty} (R_s^T - R_t^T) = t - s > 0$  whenever  $s < t$ .

3.1.3. With  $\mathbb{Q} = \mathbb{P}$  and  $B$  defined via  $B_t = \exp(t^2)$  for  $t \in \mathbb{R}_+$ , define a bond market according to Example 2.6. By construction, there exists a strictly positive supermartingale deflator. Furthermore,  $\Phi_t^\downarrow =_{\mathbb{P}} \Omega$  holds for all  $t \in \mathbb{R}_+$ , and we have  $R_t^T = T + t$  for  $t \leq T$ .

In the setting of Theorem 2.7, this example shows that  $\mathbb{P}\text{-}\lim_{T \rightarrow \infty} (R_s^T - R_t^T)$  exists and is strictly negative on  $\Phi_t^\downarrow$  for  $s < t$ . Indeed, this follows by observing that  $R_s^T - R_t^T = -(t - s) < 0$  for  $s < t$ .

We move on to the setting of Theorem 2.9. A straightforward use of (2.3) gives that, for  $0 \leq t < t' < T$ ,  $F_{t,t'}^T = T + t'$ . Pick  $s \leq t, s < s', t < t'$ ; then,  $\lim_{T \rightarrow \infty} (F_{s,s'}^T - F_{t,t'}^T) = s' - t'$ , which can take any value in  $\mathbb{R}$  for appropriate choices of  $s'$  and  $t'$ . Therefore, this example shows that we can have  $\mathbb{P}\text{-}\lim_{T \rightarrow \infty} (F_{s,s'}^T - F_{t,t'}^T) < 0$  on  $\Phi_t^\downarrow$ , if we are not working on  $\Phi_s^\uparrow$ , which shows the sharpness of the result in Theorem 2.9.

### 3.2. Optimal Rate

The rate  $O_{\mathbb{P}}^\uparrow(1/T)$  obtained in statement (2) of Theorem 2.7 cannot be improved. We shall now present an example where  $\mathbb{P}\text{-}\lim_{T \rightarrow \infty} (T(R_s^T - R_t^T))$  exists for all  $s < t$ , and is a nonzero random variable. We shall use again the construction of Example 2.6.

Consider the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ , and let  $\mathbb{Q} = \mathbb{P}$ . Let also  $W$  be a standard one-dimensional Brownian motion on the latter filtered probability space. The filtration  $(\mathcal{F}_t)_{t \in \mathbb{R}_+}$  is assumed to be the one generated by  $W$ . Let  $b \in \mathbb{R}$ . Define a short-rate process  $r$  starting at some  $r_0 \in \mathbb{R}$ , satisfying

$$r_t = e^{-t}r_0 + (1 - e^{-t})b - \sqrt{2}e^{-t} \int_0^t W_u e^u \, du + \sqrt{2}W_t$$

for all  $t \in \mathbb{R}_+$ . In differential terms it is easy to see that  $dr_t = (b - r_t)dt + \sqrt{2}dW_t$ ; this is a special case of the Vasicek model for the short rate—see Vasicek (1977). The parameters are chosen to simplify the formula (3.1) below for the yield. Let  $B := \exp(\int_0^\cdot r_t \, dt)$  and define a market according to Example 2.6. In this case, it is well known (see Brigo and Mercurio 2006) that

$$(3.1) \quad R_t^T = \frac{1 - e^{-(T-t)}}{T-t} r_t + \frac{(1 - e^{-(T-t)})^2}{2(T-t)} + (b - 1) \left( 1 - \frac{1 - e^{-(T-t)}}{T-t} \right).$$

In particular,  $\mathbb{P}\text{-}\lim_{T \rightarrow \infty} R_t^T = b - 1$  holds for all  $t \in \mathbb{R}_+$ , which implies that  $\Phi_t^\downarrow =_{\mathbb{P}} \Omega$  for all  $t \in \mathbb{R}_+$ . Using (3.1) once again we get  $\mathbb{P}\text{-}\lim_{T \rightarrow \infty} (TR_t^T - T(b - 1)) = r_t - b + 3/2$ . Therefore, for  $s < t$ ,  $\mathbb{P}\text{-}\lim_{T \rightarrow \infty} (T(R_s^T - R_t^T)) = r_s - r_t$ , which is a nontrivial Gaussian random variable.

### 3.3. Market Viability

As already discussed, asking for the existence of a strictly positive supermartingale deflator is a market viability condition that is weaker than the ones that have appeared previously in the literature. Here, we shall present an example of a market with deterministic bond prices which admits a strictly positive supermartingale deflator, but where more classical viability assumptions fail.

The probability space we are working on is left intentionally unspecified, since it plays absolutely no role. For  $0 \leq t \leq T$ , define  $P_t^T = \min\{1, \exp(1 - (T - t))\}$ . Since  $R_t^T = 1 - 1/(T - t)$  holds for  $T > t + 1$ , we obtain  $\lim_{T \rightarrow \infty} R_t^T = 1$  for all  $t \in \mathbb{R}_+$ .

Therefore,  $\Phi_t =_{\mathbb{P}} \Omega$  for all  $t \in \mathbb{R}_+$ , and the results of Theorem 2.7 and Theorem 2.9 hold trivially.

Let  $Y$  be defined via  $Y_t = \exp(-t)$  for  $t \in \mathbb{R}_+$ . Then,  $Y_t P_t^T = \min\{\exp(-t), \exp(1-T)\}$  for  $0 \leq t \leq T$ , which means that  $(Y_t P_t^T)_{t \in [0, T]}$  is a nonincreasing process, that is, a supermartingale. It follows that a strictly positive supermartingale deflator exists in this market. Therefore, there cannot exist any arbitrage of the first kind in the market if we only consider long positions in the bonds, as follows from the existence of a strictly positive supermartingale deflator in view of the general results in Kardaras (2011). However, we shall shortly see that if we allow for short positions on short-term bonds, arbitrages appear.

Let  $t \in \mathbb{R}_+$ . For any  $T \geq t + 2$ , note that

$$P_t^T = \exp(-T + t + 1) < P_t^{t+1} P_{t+1}^T = \exp(-T + t + 2).$$

Consequently, there cannot exist a probability  $\mathbb{Q}_{t, t+1}$  such that

$$P_t^T \geq P_t^{t+1} \mathbb{E}_{\mathbb{Q}_{t, t+1}}[P_{t+1}^T \mid \mathcal{F}_{t+1}].$$

Therefore, condition 2.4 in Goldammer and Schmock (2010), which already is a weaker version of existence of an EMM, is not satisfied. Furthermore, consider the following investment strategy at time  $t$ : take a long position of  $\exp(T - t - 1)$  units of a bond maturing at time  $T \geq t + 2$  and a short position in a single unit of a bond maturing at time  $t + 1$ . The capital required for this position at time  $t$  is  $-\exp(T - t - 1)P_t^T + P_t^{t+1} = -1 + 1 = 0$ . At time  $t + 1$ , the value of this position will be

$$\exp(T - t - 1)P_{t+1}^T - P_{t+1}^{t+1} = \exp(1) - 1 > 0.$$

Therefore, there exists an arbitrage in the market according to Definition 2.16 in Goldammer and Schmock (2010) once we allow for short positions on short-term bonds. Observe that one does not even need the “limiting” procedure mentioned in Goldammer and Schmock (2010) in the definition of arbitrage.

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