

Balance, growth and diversity of financial markets

Constantinos Kardaras

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Abstract A financial market comprising of a certain number of distinct companies is considered, and the following statement is proved: either a specific agent will surely beat the whole market unconditionally in the long run, or (and this “or” is not exclusive) all the capital of the market will accumulate in one company. Thus, absence of any “free unbounded lunches relative to the total capital” opportunities lead to the most dramatic failure of diversity in the market: one company takes over all other until the end of time. In order to prove this, we introduce the notion of *perfectly balanced* markets, which is an equilibrium state in which the relative capitalization of each company is a martingale under the physical probability. Then, the weaker notion of *balanced* markets is discussed where the martingale property of the relative capitalizations holds only *approximately*, we show how these concepts relate to growth-optimality and efficiency of the market, as well as how we can infer a *shadow* interest rate that is implied in the economy in the absence of a bank.

Keywords Diversity · Equivalent Martingale measure · Arbitrage · Efficiency

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0 Introduction

0.1 Discussion and results

We consider a model of a financial market that consists of d stocks of certain “distinct” companies. The distinction between companies clings on their having different risk

C. Kardaras (✉)
Mathematics and Statistics Department, Boston University,
111 Cummington Street, Boston, MA 02215, USA
e-mail: kardaras@bu.edu

and/or growth characteristics, and will find its mathematically precise definition later on in the text.

In absence of clairvoyance, the *total* capital of each company is modeled as a stochastic process S^i , $i = 1, \dots, d$. Randomness comes through a set Ω of possible *outcomes*—for each $\omega \in \Omega$ we have different realizations of $S^i(\omega)$. Financial agents decide to invest certain amounts of their wealth to different stocks, and via their actions the value of S_t^i for each time $t \in \mathbb{R}_+$ is determined.

Of major importance in our discussion will be the *distribution of market capital*, given by the *relative capitalization* $\kappa^i := S^i / (S^1 + \dots + S^d)$ of each company ($S^1 + \dots + S^d$ is the total market capital). In particular, the limiting, i.e., long-run, capital distribution will be investigated. For addressing this question, a probability \mathbb{P} is introduced that weights the different outcomes of Ω (for all events in some σ -algebra \mathcal{F}); \mathbb{P} reflects the *average* subjective feeling of the financial agents, but in this average sense it is not subjective anymore: each agent's investment decisions are fed back into the relative capitalization of the companies, and thus affects the random choice of the outcome. Via this mechanism, \mathbb{P} becomes a *real-world* probability, and can also be regarded as the subjective view of a *representative agent* in the market, whose decisions alone reflect the cumulative decisions of all “small” agents.

The time-flow of information is modeled via a *filtration* $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$. Each σ -algebra \mathcal{F}_t is supposed to include all (economical, political, etc.) information gathered up to time t and is increasing in time: $\mathcal{F}_s \subseteq \mathcal{F}_t$ for $0 \leq s < t < \infty$. A “representative agent” information structure cannot be justified, since different agents might have *very* different ability or capability to access information. This difficulty can be circumvented by choosing \mathbf{F} in a minimal way, i.e., by assuming that it is exactly the information contained in the company capitalizations—it is reasonable to assume that *every* agent has at least access to this information. This minimal information structure will turn out to be the most useful in our discussion (exactly because of its minimality property).

An important question from a modeling point of view is: *how does one go about choosing \mathbb{P} in a reasonable way in order to reflect the way financial agents act?* From the economical side, the concept of *efficiency* has been quite extensively discussed in the literature. In his famous work [Fama \(1970\)](#), Fama states that *a market in which prices “fully reflect” available information is called “efficient”*. Thus, efficiency is a property that the capitalization processes S must have under the pair (\mathbb{P}, \mathbf{F}) , but it is questionable whether it opens the door to mathematically pin down what are the possible “reasonable” probabilities \mathbb{P} .

In the field of Mathematical Finance it has been argued that a minimal condition for efficiency is absence of “free lunch” possibilities for agents; for if a free lunch existed, a sudden change in the capital distribution would occur to correct for it, which would contradict the requirement that prices fully reflect information. The notion of “no free lunch” found its mathematical incarnation in the existence of a probability \mathbb{Q} that is equivalent to \mathbb{P} (meaning that \mathbb{P} and \mathbb{Q} have the same impossibility events) under which capitalization processes *suitably deflated* have some kind of martingale property under (\mathbb{Q}, \mathbf{F}) . However, as already mentioned this is only a *minimal* condition for efficiency. Indeed, consider a two-stock market in which deflated capitalization processes are modeled by $S_t^1 = \exp(W_t^1)$ and $S_t^2 = \exp(100t + W_t^2)$ where $t \in [0, T]$ for some

finite T , and (W^1, W^2) is a 2-dimensional standard Brownian motion. An equivalent martingale measure \mathbb{Q} as described above exists for this model. Nevertheless, these being the only two investment opportunities in the market, reasonable agents would opt for the second choice over the first. Even if diversification was sought-after, significantly more capital would be held in the second rather than the first stock. This huge movement of capital would change the capitalization dynamics—this market does not appear to be in equilibrium, it is not balanced.

As mentioned previously, coupled with the choice of an equivalent martingale measure comes the choice of a *deflator* in the market. It is a usual practice to use the interest rate offered for risk-free investments for discounting. Nevertheless, it is questionable whether the interest-rate structure reflects the true market growth; a better index has to be perceived—and what would be more reasonable to use than the *total market capital*? Directly considering the *percentage* of the total capitalization that each individual company occupies, its performance in terms of the “competing” ones is assessed.

In the spirit of the above discussion, the idea of a *perfectly balanced market* is formulated by requiring that *the relative capitalizations κ^i are martingales under (\mathbb{P}, \mathbb{F})* : $\mathbb{E}[\kappa_t^i | \mathcal{F}_s] = \kappa_s^i$, for all $i = 1, \dots, d$ and $0 \leq s < t$. The last equality means that the best prediction about the future value of the relative capitalization of a company given today’s information is exactly the present value of the relative capitalization. One might ask why is this martingale property plausible. Consider, for example, what would happen if $\mathbb{E}[\kappa_t^i | \mathcal{F}_s] < \kappa_s^i$ for some company i . Since at all times the sum of all the relative capitalizations should be unit, we have $\mathbb{E}[\kappa_t^j | \mathcal{F}_s] > \kappa_s^j$ for another company j . These inequalities suggest that the overall feeling of the market is that in the future (time t) the i th company will hold on average a smaller piece of the pie than it does today (time s), with the converse holding for company j —in other words, that company i is presently overrated, while company j underrated. The reasonable thing to happen is a movement of capital from company i to company j , which would move κ_s^i downwards and κ_s^j upwards, until finally $\mathbb{E}[\kappa_t^i | \mathcal{F}_s] = \kappa_s^i$ holds for all $i = 1, \dots, d$.

Perfect balance, as an equilibrium state, can undergo much criticism: there will certainly be times at which the market “slides away” from being perfectly balanced, but it would be reasonable to assume that the market is quickly trying to readjust itself to that state (as was explained in some sense in the previous paragraph). A mathematically rigorous description of this concept would require a formulation of an “approximate martingale” property for the relative capitalization vector κ . The widely-accepted idea of assuming the existence of another probability \mathbb{Q} that is equivalent to \mathbb{P} , and such that κ is a martingale under \mathbb{Q} seems to be appropriate (actually, this exact idea has been utilized in [Yan \(1998\)](#), who has shown its equivalence to a “no-free-lunch” property relative to the total capitalization $\sum_{i=1}^d S^i$), as long as \mathbb{Q} and \mathbb{P} are “close” in some sense. This is *not* the road that will be taken here, and there are at least two good reasons: firstly, some (necessarily) ad-hoc, as well as difficult to justify in economic terms, definition of distance between \mathbb{P} and \mathbb{Q} would have to be given; secondly, existence of such a \mathbb{Q} is *not* an ω -by- ω notion (as it looks at *all* possible outcomes instead), and after all what shall be ultimately revealed is only one outcome. However, an ω -by- ω definition of plainly *balanced markets* (based on a characterization of perfectly balanced markets given by observable quantities of the model) comes to the

rescue—in some sense to be made precise later, the market is balanced if the process κ is *close* to being a martingale, but not quite there. The notion of balanced markets will turn out to be *strictly* weaker than the requirement of existence of such probability \mathbb{Q} as described above in this paragraph.

Having decomposed the state space Ω as $\Omega_b \cup \Omega_u$, where Ω_b is the set of outcomes where the market is balanced and its complement Ω_u is the set of outcomes that it is unbalanced, an analysis of the behavior of the market on each of the above two events is in order. It turns out that on Ω_u a *single* agent can beat the whole market for arbitrary levels of wealth, an unacceptable situation since the total capital of the market *should* consist of the sum of the wealths of its respective agents; on the unbalanced set this breaks down, since one *particular* agent will eventually have more capital than the whole market. It then makes sense to focus on the balanced-market outcomes Ω_b . There, it turns out that there always exists a limiting distribution of capital κ_∞ in the almost sure sense. If one further assumes that the market is *segregated*, in the sense that companies are distinct in a very weak sense, it turns out that all capital will concentrate in a *single* company. This is probably the most dramatic failure of *market diversity* pioneered by [Fernholz \(2002\)](#). In this last monograph, as well as in [Fernholz et al. \(2005\)](#), it is shown that certain diverse markets offer opportunities for free lunches relative to the market. Taking up on this, the present work shows that failure of diversity *inevitably* leads to free lunches relative to the market—at the opposite direction, non-existence of free lunches (relative to the market) a-fortiori results in the accrual of capital to one company only.

0.2 Organization of the paper

We now give a brief overview of the material.

Section 1 introduces an Itô-process model for the capitalization of companies.

Perfectly balanced markets and their characterization in terms of the drifts and volatilities of the capitalization processes are discussed in Sect. 2. To ensure a non-void discussion, abundance of perfectly balanced markets is proved.

In Sect. 3 another economically interesting equivalent formulation of perfectly balanced markets is established: they achieve maximal growth. With this characterization, we introduce implied *shadow interest rates* in the market.

Next, the concept of balanced markets (a weakening of perfectly balanced markets) is formulated in exact mathematical terms in Sect. 4. As previously noted, Ω is decomposed into Ω_b and $\Omega_u := \Omega \setminus \Omega_b$, and we characterize the balanced outcomes event Ω_b as the *maximal* set on which an agent who decides to invest according to any chosen portfolio does not have a chance to beat the market for any *unbounded* level. In other words, on Ω_b agents have a *chance* to beat the market by specific levels, but this chance is approaching zero *uniformly* over all portfolios that can be used when the level becomes arbitrarily large.

The limiting market capital distribution for balanced outcomes is taken on in Sect. 5. Existence of a limiting capital distribution κ_∞ in an almost sure sense is proved, and under a natural assumption of company segregation it is shown that all capital will concentrate in a *single* company and stay there forever.

Easy examples of a simple two-company market are presented in Sect. 6 that clarify some of the points discussed previously in the paper.

Finally, in Sect. 7 we discuss how all previous results are still valid in a more general quasi-left-continuous semimartingale environment (as opposed to a plain Itô-process one). Note that, to the best of the author's knowledge, this is the first time that results on market diversity in such a general mathematical framework are discussed; in this sense, this last section is not present just for the sake of abstract generality, but to ensure that results obtained are not sensitive to the continuous-semimartingale modeling choice.

1 The Itô-process model

A continuous semimartingale market model consisting of d different companies will be considered up to and before Sect. 7. Actually, attention will be restricted to continuous semimartingales whose drifts and covariations are absolutely continuous with respect to Lebesgue measure, Itô processes being a major example. It shall be clear later that this is done only for presentation reasons.

The *total capitalization* of each company $i = 1, \dots, d$ is denoted by S^i . These capitalizations are modeled as strictly positive stochastic processes on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$, assumed right-continuous and augmented by \mathbb{P} -null sets. The dynamics of each S^i are:

$$dS_t^i = S_t^i a_t^i dt + S_t^i dM_t^i, \quad \text{for } i = 1, \dots, d. \quad (1)$$

Here, $a := (a^1, \dots, a^d)$ is \mathbf{F} -predictable and each a^i represents the *rate of return* of each company, while $M := (M^1, \dots, M^d)$ is a (\mathbb{P}, \mathbf{F}) -local martingale for which we assume that the quadratic covariations satisfy $d[M^i, M^j]_t = c_t^{ij} dt$ for a *local covariation* symmetric matrix $c := (c^{ij})_{1 \leq i, j \leq d}$, which can be chosen \mathbf{F} -predictable—we succinctly write $d[M, M]_t = c_t dt$ in obvious notation. In order for the model (1) to make sense, a and c must satisfy

$$\int_0^t (|a_u^i| + c_u^{ii}) du < \infty, \quad \mathbb{P}\text{-a.s.}, \quad \text{for all } i = 1, \dots, d \text{ and } t \in \mathbb{R}_+.$$

Remark 1.1 Let “Leb” denote Lebesgue measure on \mathbb{R}_+ and “det” the square-matrix determinant operation. If $\mathbb{P}[\text{Leb}[\{t \in \mathbb{R}_+ \mid \det(c_t) \neq 0\}] = 0] = 1$, then there exists a standard d -dimensional (\mathbf{F}, \mathbb{P}) -Brownian motion $W \equiv (W^1, \dots, W^d)$ such that $dM_t = \langle \sigma_t, dW_t \rangle$, where σ is a square root of c : $\sigma \sigma^\top = c$ (check for example Karatzas and Shreve 1998a). Then, (1) is just an Itô process, and this model is classic—see Karatzas and Shreve (1998b). If c is degenerate on a positive $(\mathbb{P} \otimes \text{Leb})$ -measure set, the above representation is still valid if one extends the probability space. Working directly with (1) helps to avoid such complications.

Remark 1.2 The choice of “dt” above is merely for exposition purposes. At any rate, in Sect. 7 the model is generalized to the broader class of quasi-left-continuous semimartingales.

Remark 1.3 It will turn out that it is best to work under the (augmentation of the) natural filtration generated by S , which we denote by \mathbf{F}^S . Nevertheless, this restriction will *not* be imposed. Sometimes, we compare obtained results under two filtrations \mathbf{F} and \mathbf{G} , and it will be assumed that \mathbf{F} is contained in \mathbf{G} , in the sense that $\mathbf{F} \subseteq \mathbf{G}$, i.e., $\mathcal{F}_t \subseteq \mathcal{G}_t$ for all $t \in \mathbb{R}_+$. If S is a semimartingale of the form (1) under \mathbf{G} , and if $\mathbf{F} \supseteq \mathbf{F}^S$, S is also an \mathbf{F} -semimartingale and a representation of the form (1) is still valid, with the rates-of-return vector a possibly changed. (The local covariation matrix c will be the same.)

2 Perfectly balanced markets

The significance of perfectly balanced markets has already been discussed in the Introduction, so here we start directly with their definition.

Definition 2.1 The *relative capitalization* κ^i of company i is defined as

$$\kappa^i := \frac{S^i}{S^1 + \dots + S^d}, \quad \text{for } i = 1, \dots, d. \quad (2)$$

The market described by (1) will be called *perfectly balanced* with respect to the probability \mathbb{P} and the information flow \mathbf{F} if each κ^i is a (\mathbb{P}, \mathbf{F}) -martingale.

The relative capitalizations process $\kappa := (\kappa^i)_{1 \leq i \leq d}$ lives in the *open simplex*

$$\Delta^{d-1} := \left\{ x \in \mathbb{R}^d \mid 0 < x^i < 1 \text{ and } \sum_{i=1}^d x^i = 1 \right\}. \quad (3)$$

Remark 2.2 Keep the probability \mathbb{P} fixed. If the model (1) is perfectly balanced with respect some filtration \mathbf{G} that contains \mathbf{F} , which in turn contains \mathbf{F}^S , then clearly it is also perfectly balanced with respect to the information flow \mathbf{F} , since the martingale property remains. The converse does not necessarily hold: \mathbf{F} -perfect balance of the market does not imply \mathbf{G} -perfect balance: the martingale property might fail when enlarging filtrations. For agents with more information (political, insider, etc.), the market might fail to perfectly balance itself.

The weakest form of a perfectly balanced market is obtained when the filtration is \mathbf{F}^S —the one generated by S . In fact, an even smaller filtration can be used, namely, the one generated by κ (since the filtration generated by S has one extra ingredient, which is the total capitalization $\sum_{j=1}^d S^j$ that disappears when we only consider κ). It is true that one can do all subsequent work under this even smaller filtration—after all, all that we shall care about is incorporated in κ and if one starts by assuming κ is the actual capital process, everything follows.

2.1 Characterizing perfectly balanced markets

Using Itô's formula and (1), it is easily computed that for all $i = 1, \dots, d$ we have

$$d\kappa_t^i = \kappa_t^i \langle \mathbf{e}_i - \kappa_t, a_t - c_t \kappa_t \rangle dt + \kappa_t^i \langle \mathbf{e}_i - \kappa_t, dM_t \rangle, \quad (4)$$

where \mathbf{e}_i the unit vector with all zero entries but the i th, which is unit.

The above Eq. (4) for $\kappa^i, i = 1, \dots, d$ gives us a way to judge whether the market is perfectly balanced just by looking at drifts and local covariations.

Proposition 2.3 *The market is perfectly balanced if and only if there exists a predictable, one-dimensional process r with $\int_0^t |r_u| du < +\infty$ for all $t \in \mathbb{R}_+$, such that, with $\mathbf{1}$ being the vector in \mathbb{R}^d will all unit entries: $\mathbf{1} := (1, \dots, 1)$, we have:*

$$c\kappa = a - r\mathbf{1}. \quad (5)$$

Proof Each of the processes κ^i is bounded; therefore it is a martingale if and only if it is a local martingale, which by view of (4) will hold if and only if $\langle \mathbf{e}_i - \kappa, a - c\kappa \rangle = 0$. The vector processes $(\mathbf{e}_i - \kappa)_{1 \leq i \leq d}$ span the linear subspace that is orthogonal to $\mathbf{1}$. Thus, in order for κ to be a martingale there should exist a one-dimensional process r such that $a - c\kappa = r\mathbf{1}$. The fact that r can be chosen predictable and locally integrable follows from the fact that both $c\kappa$ and a have the corresponding properties.

Remark 2.4 It should be noted here that the process r plays the rôle of a *shadow interest rate* in the market, in the absence of a banking device that will produce one. To support this claim, suppose for a minute that one of the companies, say the first, behaves like a savings account, so that (if only approximately) S^1 has only a “ dt ” component, i.e., $c^{1i} = 0$ for $i = 1, \dots, d$. Multiplying from the left both sides of the relationship (5) with the unit vector \mathbf{e}_1 we get $a^1 = r$, i.e., that r is the interest rate. In the absence of a risk-free company one cannot carry the previous analysis, but an equilibrium-type argument gives the same conclusion; we come back to this point in Subsect. 3.3 with a more thorough discussion.

Remark 2.4 makes it plausible to define an *interest rate process* as being a predictable, one-dimensional process r with $\int_0^t |r_u| du < +\infty$ for all $t \in \mathbb{R}_+$.

The result of Proposition 2.3 should be interpreted as a *linear* relationship between the local covariation and the drifts of the company capitalization processes, modulo an interest rate process. It is obvious that this is a *very* restrictive condition; we shall see in Sect. 4 how to weaken it, and we shall discuss how this softer notion of (not necessarily perfectly) balanced markets ties with efficiency.

2.2 Construction of perfectly balanced markets

Equations (4) and (5) combined imply that in a perfectly balanced market the process κ must satisfy the following system of stochastic differential equations:

$$d\kappa_t^i = \kappa_t^i \langle \mathbf{e}_i - \kappa_t, dM_t \rangle, \quad \text{for all } i = 1, \dots, d. \quad (6)$$

The natural question to ask at this point is: *do mathematical models of perfectly balanced markets exist?* If they do exist, (5) as well as the stochastic differential equations (6) must hold. The following proposition shows that a plethora of perfectly balanced models exist.

Theorem 2.5 *Consider a d -dimensional continuous (\mathbf{F}, \mathbb{P}) -local martingale M whose quadratic covariation process satisfies $d[M, M]_t = c_t dt$. Then, for any \mathcal{F}_0 -measurable initial condition $\kappa_0 \equiv (\kappa_0^i)_{1 \leq i \leq d}$ with $\mathbb{P}[\kappa_0 \in \Delta^{d-1}] = 1$ the system of stochastic differential equations (6) has a unique strong solution for all $t \in \mathbb{R}_+$ that lives on Δ^{d-1} .*

Further, for any interest rate process r and \mathcal{F}_0 -measurable initial condition $S_0 \equiv (S_0^i)_{1 \leq i \leq d}$ with $S_0^i / \sum_{j=1}^d S_0^j = \kappa_0^i$ if we define $a := c\kappa + r\mathbf{1}$ and the process S via (1), we get a model of a perfectly balanced market.

Proof The second paragraph of the Proposition's statement is obvious from our previous discussion; we only need prove that the system of stochastic differential equations (6) has a unique strong solution for $t \in \mathbb{R}_+$ that lives on Δ^{d-1} .

To begin, consider the unit cube $[0, 1]^d$ in \mathbb{R}^d . The volatility coefficients appearing in (6) are quadratic in κ , thus are obviously Lipschitz as a functions of κ on $[0, 1]^d$; then, the standard theorem on strong solutions of stochastic differential equations gives that (6) has a unique strong solution for t in a stochastic interval $[[0, \tau]]$, where τ is a stopping time such that for all $t < \tau$ we have $\kappa_t \in (0, 1)^d$, while on $\{\tau < +\infty\}$ we have $\kappa_\tau \in \partial[0, 1]^d$ (the boundary of $[0, 1]^d$). First, it will be shown that κ is Δ^{d-1} -valued on $[[0, \tau]]$, and then that $\mathbb{P}[\tau = +\infty] = 1$.

Using (6) one can compute that on $[[0, \tau]]$ the process $z := 1 - \langle \mathbf{1}, \kappa \rangle$ satisfies the stochastic differential equation $dz_t = -z_t \langle \kappa_t, dM_t \rangle$ (observe that now κ is known). Since $z_0 = 0$, the unique strong solution of this last equation is $z \equiv 0$, so $\langle \mathbf{1}, \kappa \rangle = 1$ and κ is Δ^{d-1} -valued on $[[0, \tau]]$.

Now, on $[[0, \tau]]$ we have $0 < \kappa^i < 1$ for each $i = 1, \dots, d$. Using Itô's formula for the logarithmic function and (6) once again we get for $t \in [[0, \tau]]$ that

$$\log \kappa_t^i = \log \kappa_0^i - \frac{1}{2} \int_0^t \langle \mathbf{e}_i - \kappa_u, c_u(\mathbf{e}_i - \kappa_u) \rangle du + \int_0^t \langle \mathbf{e}_i - \kappa_u, dM_u \rangle.$$

Both the finite-variation part and the quadratic variation of the local martingale part of the semimartingale $\log \kappa^i$ are finite on any bounded interval as long as $\kappa \in \Delta^{d-1}$; it follows that on the event $\{\tau < +\infty\}$ we have $\lim_{t \uparrow \tau} \log \kappa_t^i \in \mathbb{R}$, which implies that $\lim_{t \uparrow \tau} \kappa_t^i > 0$. Since κ is Δ^{d-1} -valued on $[[0, \tau]]$, it also follows that $\lim_{t \uparrow \tau} \kappa_t^i < 1$ for all $i = 1, \dots, d$. This contradicts the fact that we are assumed to work on the event $\{\tau < +\infty\}$, therefore $\mathbb{P}[\tau = +\infty] = 1$.

Remark 2.6 One of the reasons *not* to require \mathbf{F} to be the one generated by S is the constructive Theorem 2.5, where we start a-priori with some filtration \mathbf{F} that makes M a \mathbb{P} -martingale and r adapted. If wanted, after the construction of κ has been carried out we can pass from \mathbf{F} to the generally smaller \mathbf{F}^S .

Remark 2.7 Apart from its mathematical significance, Theorem 2.5 also has interesting economic implications. When writing the dynamics (1) of a model we assume that

both the drift vector a and the local covariation matrix c are observable. Nevertheless, both in a statistical and in a philosophical sense, covariances are easier to assess than drifts. From a statistical point of view, high-frequency data can lead to reasonably good estimation of c —and the ideal case of continuously collected data leads to perfect estimation. Nevertheless, there is no easy way to estimate a , even if we assume it is a constant: one has to wait for too long a time to get any sensible estimate. In a more philosophical sense, economic agents might not have a complete sense of how the prices will move, but they might very well have an idea of how risky the companies are, and how a change in the capitalization of one company would affect another one, i.e., exactly the local covariation matrix c . To this effect, Theorem 2.5 implies that simple knowledge of the local covariations c , the interest rate r (see Remark 2.4 and Subsect. 3.3 in this respect) and the relative capitalizations at time $t = 0$ is enough to provide the whole process of relative capitalizations; and by this, we also get the drifts a . Thus, in perfectly balanced markets, a good estimate of c is enough to provide good estimates for the drift a as well.

3 Growth-optimality of perfectly balanced markets

We discuss here an “economically optimal” property of perfectly balanced markets that actually turns out to be an equivalent formulation in a sense. We also elaborate on how the process r of Proposition 2.3 should be thought as a shadow interest rate prevailing in the market.

3.1 Agents and investment

In a market with d companies whose capitalizations are described by the dynamics (1), we also consider a *savings account* offered by a bank, described by some interest rate process r . One unit of currency invested in (i.e., loaned to) the bank at time s will grow to $\int_s^t r_u du$ by time $t > s$. We remark that existence of a bank does not add wealth to the market directly, although can do so indirectly by adding more flexibility to the financial agents—in other words, the net amount invested in the bank must be zero: some lend and some borrow, but the total position should be neutral.

We now discuss the behavior of an individual agent in the market; this agent decides to invest a portion of the total capital-in-hand in each of the d companies, and the remaining wealth in the savings account. We shall be denoting by π_t^i the proportion of the capital invested in the i th company; then, $1 - \langle \pi, \mathbf{1} \rangle$ proportion of the capital-in-hand is put into savings. To ensure than no clairvoyance into the future is allowed, the vector process $\pi := (\pi^i)_{1 \leq i \leq d}$ should be predictable with respect to the filtration of the individual agent, which is at least as large as \mathbf{F}^S .

We model portfolio constraints that an agent might be faced with via a set-valued process \mathfrak{C} ; henceforth we shall be assuming that for each $(\omega, t) \in \Omega \times \mathbb{R}_+$:

1. $\overline{\Delta}^{d-1} \subseteq \mathfrak{C}(\omega, t)$, where $\overline{\Delta}^{d-1}$ is the closure of the open simplex of (3);
2. the set $\mathfrak{C}(\omega, t)$ is closed and convex; and

3. \mathfrak{C} is predictable, in the sense that $\{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid \mathfrak{C}(\omega, t) \cap F \neq \emptyset\}$ is a predictable set for all closed $F \subseteq \mathbb{R}^d$.

Then, a \mathfrak{C} -constrained portfolio is a predictable, d -dimensional process π that satisfies $\pi(\omega, t) \in \mathfrak{C}(\omega, t)$ for all $(\omega, t) \in \Omega \times \mathbb{R}_+$, and

$$\int_0^t (|\langle \pi_u, a_u \rangle| + \langle \pi_u, c_u \pi_u \rangle) du < \infty, \quad \text{for all } t \in \mathbb{R}_+. \quad (7)$$

The class of all \mathfrak{C} -constrained portfolios is denoted by $\Pi_{\mathfrak{C}}$.

The most important case in the discussion to follow is the most restrictive case of constraints $\mathfrak{C} = \overline{\Delta}^{d-1}$: the agent has only access to invest in the “actual” companies of the market—in this case, the bank is not even needed.

The portfolio integrability requirement (7) is a technical one, but it is the weakest assumption in order for the stochastic integrals appearing below in (8) to make sense. The requirement is certainly satisfied if π is \mathbb{P} -a.s. bounded on every interval $[0, t]$ for $t \in \mathbb{R}_+$ —for example if π is $\overline{\Delta}^{d-1}$ -valued.

The initial investment of an agent at time zero is always normalized to be a unit of currency. Assuming this and investing according to $\pi \in \Pi_{\mathfrak{C}}$, the corresponding wealth process V^π of the particular agent is described by $V_0^\pi = 1$ and

$$\frac{dV_t^\pi}{V_t^\pi} = \sum_{i=1}^d \pi_t^i \frac{dS_t^i}{S_t^i} + \left(1 - \sum_{i=1}^d \pi_t^i\right) r_t dt = (r_t + \langle \pi_t, a_t - r_t \mathbf{1} \rangle) dt + \langle \pi_t, dM_t \rangle. \quad (8)$$

The collective investment of all agents is captured by the percentage of the total market capitalization invested in each company, i.e., the relative capitalizations $\kappa = (\kappa^i)_{1 \leq i \leq d}$, which is an \mathbf{F} -predictable vector process (as it is \mathbf{F}^S -adapted and continuous) and satisfies $\kappa \in \Delta^{d-1}$; thus κ can be viewed as a portfolio, and as such it is called the *market portfolio*. Here is the reason for such a name: using (8) one checks that $V^\kappa = \langle S, \mathbf{1} \rangle / \langle S_0, \mathbf{1} \rangle$, where $\langle S, \mathbf{1} \rangle = \sum_{i=1}^d S^i$ is the total capital of the market: investing according to κ is tantamount to owning the whole market, relative to the initial investment, which is normalized to unit.

3.2 Growth and growth-optimality

The process $a^\pi := \langle \pi, a \rangle$ appearing in (8) is known as the *rate of return* of V^π ; it is the instantaneous return that the strategy gives on the invested capital. Nevertheless, for long-time-horizon investments, rates of return fail to give a good idea of the behavior of the wealth process. A more appropriate tool for analyzing asymptotic behavior is the growth rate (see for example [Fernholz 2002](#)), which we now define.

For a portfolio $\pi \in \Pi_{\mathcal{C}}$, its log-wealth process is the semimartingale $\log V^{\pi}$. Itô's formula gives $d \log V_t^{\pi} = g_t^{\pi} dt + \langle \pi_t, dM_t \rangle$, where

$$g^{\pi} := r + \langle \pi, a - r\mathbf{1} \rangle - \frac{1}{2} \langle \pi, c\pi \rangle \quad (9)$$

is the *growth rate* of the portfolio π . The portfolio $\rho \in \Pi_{\mathcal{C}}$ will be called *growth-optimal in the \mathcal{C} -constrained class* if

$$g^{\rho}(\omega, t) = g^*(\omega, t) := \sup_{\pi \in \mathcal{C}} g^{\pi}(\omega, t), \text{ for all } (\omega, t) \in \Omega \times \mathbb{R}_+. \quad (10)$$

The whole market is called a *growth market* if the market portfolio κ is growth optimal over all possible portfolios.

Proposition 3.1 *A market described by an interest rate process r and (1) for the company capitalizations is a growth market if and only if $c\kappa = a - r\mathbf{1}$.*

Proof In order to have a growth market, κ must solve the quadratic problem

$$\max_p \left\{ r + \langle p, a - r\mathbf{1} \rangle - \frac{1}{2} \langle p, cp \rangle \right\} \quad (11)$$

over all $p \in \mathbb{R}^d$ where we have hidden the dependence on (ω, t) . The growth rate function of (9) is concave, and first-order conditions imply that in order for κ to be a solution to the optimization problem we must have $a - c\kappa = r\mathbf{1}$. \square

Remark 3.2 Generalizing a bit the method-of-proof of Proposition 3.1, we can give the following characterization: ρ is \mathcal{C} -constrained growth optimal portfolio if and only if V^{π} / V^{ρ} is a supermartingale for all $\pi \in \Pi_{\mathcal{C}}$. Indeed, for any two portfolios π and ρ , one can use (8) and Itô's formula to get that V^{π} / V^{ρ} is a supermartingale if and only if $\langle \pi - \rho, a - r\mathbf{1} - c\rho \rangle \leq 0$; this is exactly the first-order condition for maximization of (11) over \mathcal{C} .

Remark 3.3 Statistical tests of the “perfectly balanced market” hypothesis have appeared in the literature in the seventies, where it was actually tested whether the market portfolio is equal to the growth-optimal one (the connection is obvious in view of Proposition 3.1—see also the discussion in the next Subsect. 3.3). We mention in particular the works of Roll (1973), as well as Fama and MacBeth (1974) that treat the New York Stock Exchange as the “market”. In both papers, there does not seem to be conclusive evidence on whether the perfect-balance hypothesis holds or not; although it cannot be rejected at any reasonably high statistical significance level, there are noteworthy deviations mentioned therein.

3.3 Interest rate

Proposition 3.1 clearly shows the connection between growth and perfectly balanced markets. The difference between Propositions 2.3 and 3.1 is that in the former we *infer*

the existence of an interest rate r that satisfies $c\kappa = a - r\mathbf{1}$, while in the latter the interest rate process is given as a market parameter.

In fact, if existence of an interest rate process is not assumed, and a growth market is defined as one where κ maximizes the growth rate over all portfolios in the constrained set $\mathfrak{C} = \{x \in \mathbb{R}^d \mid \langle x, \mathbf{1} \rangle = 1\}$, then going through the proof of Proposition 3.1 using Lagrange-multiplier theory for constrained optimization, the relationship $c\kappa = a - r\mathbf{1}$ for *some* interest rate process r will be inferred again, exactly as in the case of perfectly balanced markets. Thus, the two concepts of growth and perfectly balanced markets are identical in this sense.

Now, an equilibrium argument will be used to show that even in the absence of a bank, the arbitrary process r obtained in the case where the market is perfectly balanced really plays a rôle of an interest rate. Suppose that all of a sudden, the market decides to build a bank and has to decide on what interest rate \tilde{r} to offer. In the next paragraph we answer the following question: *What should this process \tilde{r} be in order for the market to stay in perfectly balanced state?* Then, \tilde{r} is an equilibrium interest rate.

Before the introduction of a bank the market was perfectly balanced, i.e., $c\kappa = a - r\mathbf{1}$ was true for *some* one-dimensional process r . The introduction of a savings account gives more freedom to individual agents: now they can borrow or lend at the risk-free interest rate \tilde{r} . The “representative agent” in the augmented (with the bank) market will still try to maximize growth, as before, and for this representative agent the wealth proportion held in the bank should be zero. Indeed, if in trying to maximize the growth rate the representative agent found that the optimal holdings in the risk-free security is positive, the overall feeling of the agents is that the interest rate level \tilde{r} is attractive for saving, and more agents would be inclined to save money than to borrow for investment in the riskier company of the market; this would create instability because supply for funds to be invested in riskier companies would exceed demand. The exact opposite of what was just described would happen if the representative agent’s optimal holdings in the risk-free security were negative. Proposition 3.1 implies that after the introduction of a bank we should have $c\kappa = a - \tilde{r}\mathbf{1}$; nevertheless, just before the bank appeared we had $c\kappa = a - r\mathbf{1}$. The only way that both can hold is $r = \tilde{r}$, which shows that r really plays the rôle of an equilibrium interest rate process, even in the absence of a bank.

4 Balanced markets

The definition of a perfectly balanced market is restrictive, since the martingale property for the relative capitalizations is not expected to exactly hold. Sometimes it might fail and it also might take some time to return to equilibrium, as explained in the Introduction. We therefore want to say that the market will be balanced (though not necessarily perfectly) if the martingale property holds only “approximately”. No such reasonable notion exists, and one needs to work around it. In this section we elaborate on balanced markets and their close relation to a concept of “efficiency”.

4.1 Formal definitions

According to Proposition 3.1 and the content of Subsect. 3.3, a market equipped with a bank is perfectly balanced if and only if $g^k = g^*$, where $g^* \equiv g^{*(F, \mathcal{C})}$ is the maximal growth that can be obtained by using F -predictable and \mathcal{C} -constrained portfolios. In general, we have $g^k \leq g^*$, and the market will be balanced if this difference is not very large.

Definition 4.1 For some filtration F and constraints set \mathcal{C} , define the continuous and increasing *loss-of-perfect-balance* process L via

$$L_t \equiv L_t^{F, \mathcal{C}} := \int_0^t (g_u^{*(F, \mathcal{C})} - g_u^k) du,$$

and write $\Omega = \Omega_b \cup \Omega_u$, where $\Omega_b \equiv \Omega_b^{F, \mathcal{C}} := \{L_\infty^{F, \mathcal{C}} < +\infty\}$ are the *balanced* outcomes and $\Omega_u \equiv \Omega_u^{F, \mathcal{C}} := \{L_\infty^{F, \mathcal{C}} = +\infty\} = \Omega \setminus \Omega_b$ the *unbalanced* outcomes.

If $\mathbb{P}[\Omega_b^{F, \mathcal{C}}] = 1$, the market described by (1) will be called *balanced* with respect to the probability \mathbb{P} , the information flow F and the constraints \mathcal{C} .

Remark 4.2 If a predictable process ρ that solves the maximization problem (10) exists for all $(\mathbb{P} \otimes \text{Leb})$ -almost every $(\omega, t) \in \Omega \times \mathbb{R}_+$ and ρ satisfies the integrability conditions (7) we then have $g^* = g^\rho$. This always happens if \mathcal{C} is contained in a fixed compact subset K of \mathbb{R}^d for all $(\omega, t) \in \Omega \times \mathbb{R}_+$.

In general, a predictable process ρ solving (10) might not exist; even if it does exist, the integrability conditions (7) might not be fulfilled. It can be shown that ρ exists and satisfies (7) if and only if $L_t < \infty$ for all $t \in \mathbb{R}_+$, \mathbb{P} -a.s. A thorough discussion of these points is made in [Karatzas and Kardaras \(2007\)](#).

Consider two filtrations F and G such that $F^S \subseteq F \subseteq G$; G -perfect balance implies F -perfect balance. The same holds for simply balanced markets.

Proposition 4.3 Consider two pairs of filtrations and constraints (F, \mathcal{C}) and (G, \mathcal{K}) with $F^S \subseteq F \subseteq G$ and $\mathcal{C} \subseteq \mathcal{K}$. We then have $g^{*(F, \mathcal{C})} \leq g^{*(G, \mathcal{K})}$; as a consequence, $L^{F, \mathcal{C}} \leq L^{G, \mathcal{K}}$ and $\Omega_b^{F, \mathcal{C}} \subseteq \Omega_b^{G, \mathcal{K}}$.

Proof For all $n \in \mathbb{N}$ set $\mathcal{C}_n := \mathcal{C} \cap [-n, n]^d$ and $\mathcal{K}_n := \mathcal{K} \cap [-n, n]^d$. We then have that $\lim_{n \rightarrow \infty} \uparrow g^{*(F, \mathcal{C}_n)} = g^{*(F, \mathcal{C})}$ and $\lim_{n \rightarrow \infty} \uparrow g^{*(G, \mathcal{K}_n)} = g^{*(G, \mathcal{K})}$ and thus it suffices to prove $g^{*(F, \mathcal{C})} \leq g^{*(G, \mathcal{K})}$ under the assumption $\mathcal{C} \subseteq \mathcal{K} \subseteq K$ for some compact set K . According to Remark 4.2, under this assumption the growth-optimal portfolios $\rho(F, \mathcal{C})$ and $\rho(G, \mathcal{K})$ exist and $g^{*(F, \mathcal{C})} = g^{\rho(F, \mathcal{C})}$ as well as $g^{*(G, \mathcal{K})} = g^{\rho(G, \mathcal{K})}$. From Remark 3.2 we know that $V^{\rho(F, \mathcal{C})}/V^{\rho(G, \mathcal{K})}$ is a positive supermartingale, which gives that $\log(V^{\rho(F, \mathcal{C})}/V^{\rho(G, \mathcal{K})})$ is a local supermartingale; the drift of the last local supermartingale—which should be decreasing—is $\int_0^{\cdot} (g_t^{\rho(F, \mathcal{C})} - g_t^{\rho(G, \mathcal{K})}) dt$, which gives us $g^{*(F, \mathcal{C})} \leq g^{*(G, \mathcal{K})}$ and completes the proof.

4.2 Some discussion

We contemplate slightly on balanced markets.

4.2.1 Trivial example

Perfectly balanced markets satisfy $L \equiv 0$, and are therefore balanced.

4.2.2 No bank

Let us assume now that $\mathfrak{C} = \{x \in \mathbb{R}^d \mid \langle x, \mathbf{1} \rangle = 1\}$ —we are allowed to invest in the risky companies, but there is no bank (for us, at least).

We assume that c is non-degenerate for $(\mathbb{P} \otimes \text{Leb})$ -almost every $(\omega, t) \in \Omega \times \mathbb{R}_+$; then, the maximization problem (10) has a solution ρ that satisfies $c\rho = a - r\mathbf{1}$ for some *unique* one-dimensional process r . On the $(\mathbb{P} \otimes \text{Leb})$ -full measure subset of $\Omega \times \mathbb{R}_+$ where c is non-singular it is clear that $\rho = c^{-1}(a - r\mathbf{1})$; using $\langle \rho, \mathbf{1} \rangle = 1$ it is easy to see that

$$r = \frac{\langle a, c^{-1}\mathbf{1} \rangle - 1}{\langle \mathbf{1}, c^{-1}\mathbf{1} \rangle} \quad (12)$$

Now, straightforward computations give

$$\mathfrak{g}^* - \mathfrak{g}^\kappa \equiv \mathfrak{g}^\rho - \mathfrak{g}^\kappa = \frac{1}{2} \langle \kappa - \rho, c(\kappa - \rho) \rangle = \frac{1}{2} \left| c^{-1/2}(c\kappa - a + r\mathbf{1}) \right|^2.$$

(One can also show the last relationship observing that V^κ / V^ρ is a local martingale and taking the logarithm.) Perfectly balanced markets satisfy $c\kappa - a + r\mathbf{1} = 0$ identically with r given by (12); simply balanced markets do not satisfy the last equation identically, but approximately: $\int_0^\infty |c_t^{-1/2}(c_t\kappa_t - a_t + r_t\mathbf{1})|^2 dt < \infty$.

We remark that on Ω_b , r earns the name of an interest rate process, i.e., it is locally integrable. More specifically, it will be shown below that for any random time τ we have $\int_0^\tau |r_u| du < \infty$ on $\{\tau < \infty, L_\tau < \infty\}$. Define $F_t := \int_0^t \langle \kappa_u, c_u \kappa_u \rangle du$; on $\{\tau < \infty\}$ we have $F_\tau < \infty$. The Cauchy–Schwartz inequality gives

$$\int_0^\tau |\langle \kappa_u, c_u \kappa_u \rangle - \langle \kappa_u, a_u \rangle - r_u| du = \int_0^\tau |\langle \kappa_u, c_u(\kappa_u - \rho_u) \rangle| du \leq \sqrt{L_\tau F_\tau} < \infty,$$

on $\{\tau < \infty, L_\tau < \infty\}$. Since on $\{\tau < \infty\}$ we have $\int_0^\tau |\langle \kappa_u, c_u \kappa_u \rangle - \langle \kappa_u, a_u \rangle| du < \infty$; we conclude that $\int_0^\tau |r_u| du < \infty$ on $\{\tau < \infty, L_\tau < \infty\}$, as proclaimed.

4.2.3 Interest rate revisited

Continuing the above discussion, where no bank is present, suppose that we wish to introduce an interest rate process \tilde{r} in such a way as to keep the market balanced—at least on the event that it was balanced before. In the case of perfectly balanced

market, $\tilde{r} \equiv r$ must hold—here, we shall see that we have this last equality holding *approximately*.

We still assume that c is non-singular on a set of full $(\mathbb{P} \otimes \text{Leb})$ -measure (which is very reasonable to justify the introduction of a bank). A solution $\tilde{\rho}$ of the optimization problem (10) in the market augmented with the bank exists, and $c\tilde{\rho} = a - \tilde{r}\mathbf{1}$. Straightforward, but somewhat lengthy, computations show that

$$\int_0^\infty (\mathfrak{g}_t^{\tilde{\rho}} - \mathfrak{g}_t^\kappa) dt = \int_0^\infty (\mathfrak{g}_t^\rho - \mathfrak{g}_t^\kappa) dt + \frac{1}{2} \int_0^\infty \langle \mathbf{1}, c_t^{-1} \mathbf{1} \rangle |\tilde{r}_t - r_t|^2 dt,$$

where $\rho := c^{-1}(a - r\mathbf{1})$ and r is given by (12). Introducing a bank that offers interest rate \tilde{r} keeps the market balanced if and only if $\int_0^\infty \langle \mathbf{1}, c_t^{-1} \mathbf{1} \rangle |\tilde{r}_t - r_t|^2 dt < \infty$, which can be seen as an approximate equality between r and \tilde{r} .

4.2.4 Equivalent Martingale measures

We now delve into the relationship between balanced markets and the existence of a probability $\mathbb{Q} \sim \mathbb{P}$ that makes the relative capitalizations κ^i \mathbb{Q} -martingales. We call such a probability \mathbb{Q} an equivalent martingale measure (EMM), although it does not apply directly to the *actual*, rather to the *relative* capitalizations. The concept of a balanced market is closely related, but weaker than the existence of an EMM. It is not hard to see why it is weaker: assume the existence of an EMM \mathbb{Q} and denote by Z the density process, i.e., $Z_t := (d\mathbb{Q}/d\mathbb{P})|_{\mathcal{F}_t}$. Since $\mathbb{Q} \sim \mathbb{P}$, we have $\mathbb{P}[Z_\infty] > 0$. The Kunita–Watanabe decomposition implies

$$Z_t = E_t N_t, \text{ with } E_t := \exp \left(\int_0^t \langle h_u, dM_u \rangle - \frac{1}{2} \int_0^t \langle h_u, c_u h_u \rangle du \right)$$

where h is an d -dimensional predictable process and the strictly positive local martingale N is strongly orthogonal to M . The integrand h need not be unique, but the local martingale $\int_0^\cdot \langle h_u, dM_u \rangle$ is. Since κ has to be a \mathbb{Q} -martingale, one can show that we can choose $h = \rho - \kappa$, where ρ is the growth-optimal portfolio, that *must* exist. Since $Z_\infty > 0$ and $N_\infty < +\infty$, \mathbb{P} -a.s. we also have that $E_\infty > 0$, \mathbb{P} -a.s.; in view of Lemma A.2 this is equivalent to saying that the quadratic variation of the local martingale $\int_0^\cdot \langle h_u, dM_u \rangle$ is finite at infinity—but this is exactly L_∞ ; thus the existence of an EMM implies that the market is balanced.

In Sect. 6 we shall see by example that the notion of a balanced market is actually *strictly* weaker than existence of an EMM \mathbb{Q} .

4.3 Balanced markets and efficiency

The chances for an agent to do well relatively to the overall wealth are very different depending on which of the events Ω_b and Ω_u is being considered. The next result gives a characterization of Ω_u in terms of beating the whole market.

Theorem 4.4 *We consider the model (1) valid under some filtration $\mathbf{F} \supseteq \mathbf{F}^S$ and the constraints set $\mathfrak{C} = \overline{\Delta}^{d-1}$.*

- *On Ω_b , and for any portfolio $\pi \in \Pi_{\mathfrak{C}}$ the limit $\lim_{t \rightarrow \infty} (V_t^\pi / V_t^\kappa)$ of the relative wealth process exists and is \mathbb{R}_+ -valued. The probability of beating the whole market for ever-increasing levels converges to zero uniformly among all portfolios:*

$$\lim_{m \rightarrow \infty} \downarrow \sup_{\pi \in \Pi_{\mathfrak{C}}} \mathbb{P} \left[\sup_{t \in \mathbb{R}_+} \left(\frac{V_t^\pi}{V_t^\kappa} \right) > m \mid \Omega_b \right] = 0. \quad (13)$$

- *Further, Ω_b is the maximal set that (13) holds: there exists $\rho \in \Pi_{\mathfrak{C}}$ such that $\lim_{t \rightarrow \infty} (V_t^\rho / V_t^\kappa)$ is $(0, \infty]$ -valued, and $\Omega_u = \{\lim_{t \rightarrow \infty} (V_t^\rho / V_t^\kappa) = \infty\}$.*

Proof Consider the growth optimal portfolio ρ in the class $\Pi_{\mathfrak{C}}$ —since \mathfrak{C} is a constant compact subset of \mathbb{R}^d this certainly exists. Take now any portfolio $\pi \in \Pi_{\mathfrak{C}}$; Remark 3.2 gives that the relative wealth process V^π / V^ρ is a positive supermartingale. Then, for any $l > 0$ we have $\mathbb{P}[\sup_{t \in \mathbb{R}_+} (V_t^\pi / V_t^\rho) > l] \leq l^{-1}$, i.e., the collection $\{\sup_{t \in \mathbb{R}} (V_t^\pi / V_t^\rho)\}_{\pi \in \Pi_{\mathfrak{C}}}$ is bounded in probability. Further, Itô's formula for the semimartingale $\log(V^\kappa / V^\rho)$ reads

$$\log \frac{V_t^\kappa}{V_t^\rho} = -L_t + \int_0^t \langle \kappa_u - \rho_u, dM_u \rangle. \quad (14)$$

Observe then that on Ω_b both the finite-variation part and the quadratic variation of the local martingale part of the semimartingale $\log(V^\kappa / V^\rho)$ are finite all the way to infinity, thus $\inf_{t \in \mathbb{R}_+} (V_t^\kappa / V_t^\rho) \in (0, +\infty)$. Writing $V^\pi / V^\kappa = (V^\pi / V^\rho)(V^\rho / V^\kappa)$ for all $\pi \in \Pi_{\mathfrak{C}}$, we see that the collection $\{\sup_{t \in \mathbb{R}_+} (V_t^\pi / V_t^\kappa)\}_{\pi \in \Pi_{\mathfrak{C}}}$ is bounded in probability on Ω_b , which is exactly the first claim (13).

The fact that L dominates twice the quadratic variation process of the local martingale $\int_0^{\cdot} \langle \kappa_u - \rho_u, dM_u \rangle$ enables one to use the strong law of large numbers (Lemma A.1 of the Appendix) in (14) and show that we have

$$\lim_{t \rightarrow \infty} \frac{\log(V_t^\rho / V_t^\kappa)}{L_t} = 1,$$

on $\Omega_u = \{L_\infty = \infty\}$, which proves the second claim.

Remark 4.5 The assumption $\mathfrak{C} = \overline{\Delta}^{d-1}$ in Theorem 4.4 is being made to ease the proof, and also because it will be the only case we need in the sequel. This assumption can be dropped; Theorem 4.4 still holds, with some possible slight changes which we now describe.

The essence of the assumption $\mathfrak{C} = \overline{\Delta}^{d-1}$ was to make sure that the growth-optimal portfolio ρ exists in the class $\Pi_{\mathfrak{C}}$; thus, the proof remains valid whenever \mathfrak{C} is contained in a compact subset of \mathbb{R}^d . In the general case, one might not be able to use ρ directly (since it might not even exist), but rather a subsequence of $(\rho_n)_{n \in \mathbb{N}}$ where ρ_n defined to be the \mathfrak{C}_n -constrained growth-optimal portfolio where $\mathfrak{C}_n := \mathfrak{C} \cap [-n, n]^d$ and replace the second bullet in Theorem 4.4 by

- Further, Ω_b is the maximal set that (13) holds: if $\mathbb{P}[\Omega_u] > 0$ one can find a sequence of portfolios $(\rho_n)_{n \in \mathbb{N}}$ such that $\lim_{t \rightarrow \infty} (V_t^{\rho_n} / V_t^\kappa)$ exists and

$$\lim_{n \rightarrow \infty} \mathbb{P} \left[\lim_{t \rightarrow \infty} \frac{V_t^{\rho_n}}{V_t^\kappa} > n \mid \Omega_u \right] = 1.$$

5 Segregation and limiting capital distribution of balanced markets

Here, we describe the limiting behavior of the market on the set of balanced outcomes Ω_b . We take the latter event to be as large as possible, which by Proposition 4.3 means that for this section we consider the case where the filtration is \mathbf{F}^S and $\mathfrak{C} = \overline{\Delta}^{d-1}$. By Theorem 4.4, on the event Ω_u an investor with minimal information can construct an all-long portfolio that can beat the market unconditionally; to keep our sanity, it is best to assume that the market is balanced.

5.1 Limiting capital distribution

The following result is a simple corollary of Theorem 4.4. (All set-inclusions appearing from now on are valid modulo \mathbb{P} .)

Proposition 5.1 $\Omega_b \subseteq \{\kappa_\infty := \lim_{t \rightarrow \infty} \kappa_t \text{ exists}\}$.

Proof Write $\kappa^i = \kappa^i_0 (V^{e_i} / V^\kappa)$ and use the first claim of Theorem 4.4 with $\pi = e_i$. \square

Thus, we know that on Ω_b there exists a limiting capital distribution in a very strong sense: there is almost sure convergence of the relative capitalizations vector. The next task is to identify this distribution.

5.2 Sector equivalence and segregation

We give below a definition of some sort of distance between two companies. To introduce the definition and get an idea of what it means, remember that if π_1 and π_2 are two portfolios, the drift of the log-wealth process $\log(V^{\pi_1} / V^{\pi_2})$ is $\int_0^{\cdot} \mathfrak{g}_t^{\pi_1 | \pi_2} dt$, where $\mathfrak{g}^{\pi_1 | \pi_2} := \mathfrak{g}^{\pi_1} - \mathfrak{g}^{\pi_2}$, and that its quadratic variation is $\int_0^{\cdot} c_t^{\pi_1 | \pi_2} dt$ where $c^{\pi_1 | \pi_2} := \langle \pi_2 - \pi_1, c(\pi_2 - \pi_1) \rangle$. In the case where the portfolios are unit vectors e_i, e_j for some $1 \leq i, j \leq d$ we write $\mathfrak{g}^{i | j}$ and $c^{i | j}$ for $\mathfrak{g}_{e_i | e_j}$ and $c^{e_i | e_j}$, respectively.

Definition 5.2 Say that two companies i and j in the market are *equivalent* (on the outcome ω) and denote $i \sim j$ (more precisely $i \sim_\omega j$) if their total distance

$$d^{i|j} := \int_0^\infty \left(|\mathbf{g}_t^{i|j}| + \frac{1}{2} c_t^{i|j} \right) dt, \quad (15)$$

satisfies $d^{i|j}(\omega) < \infty$, and write $i \approx j$ ($i \approx_\omega j$ is more precise) if $d^{i|j}(\omega) = \infty$.

The *segregation* event is $\Sigma := \{i \approx j, \text{ for all pairs of companies } (i, j)\}$; if $\mathbb{P}[\Sigma] = 1$, the market will be called *segregated*.

Market segregation is conceptually very natural. Indeed, if two companies satisfy $i \sim_\omega j$ for some outcome $\omega \in \Omega$, then the total quadratic variation of the difference of their returns all the way to infinity is finite; in this sense, the total cumulative uncertainty (up to infinity) that they bear is *very comparable*. The same happens for their growth rates, as (15) implies. In this case they should really be viewed and modeled as the same entity of the market. To really speak of “different” companies, they must have some different uncertainty *or* growth characteristics; this makes Definition 5.2 perfectly reasonable.

Remark 5.3 An equivalence relation between portfolios π_1 and π_2 can similarly be defined, by postulating that $\pi_1 \sim_\omega \pi_2$ if $\int_0^\infty (|\mathbf{g}_t^{\pi_1|\pi_2}| + c_t^{\pi_1|\pi_2}/2) dt < \infty$. Then, we can write $\Omega_b = \{\rho \sim \kappa\}$. To wit, remember that $\Omega_b = \{\int_0^\infty \mathbf{g}_t^{\rho|\kappa} dt < \infty\}$, so we certainly have $\{\rho \sim \kappa\} \subseteq \Omega_b$. On the other hand, since V^κ / V^ρ is a supermartingale, it is easy to see that we have $2\mathbf{g}_t^{\rho|\kappa} \geq c_t^{\rho|\kappa}$, which gives $\{\int_0^\infty \mathbf{g}_t^{\rho|\kappa} dt < \infty\} \subseteq \{\int_0^\infty c_t^{\rho|\kappa} dt < \infty\}$, and thus $\Omega_b = \{\rho \sim \kappa\}$.

It should be clear that

$$\{i \sim j\} \subseteq \left\{ \lim_{t \rightarrow \infty} \left(\log \frac{\kappa_t^i}{\kappa_t^j} \right) \text{ exists} \right\} = \left\{ \lim_{t \rightarrow \infty} \frac{\kappa_t^i}{\kappa_t^j} \text{ exists and is strictly positive} \right\}. \quad (16)$$

Remark 5.4 The relationship \sim of Definition 5.2 is an equivalence relationship. Indeed, suppose that i, j and k are three companies. That $i \sim i$ is evident, since $\mathbf{g}^{i|i} = c^{i|i} = 0$; also, $i \sim j \Leftrightarrow j \sim i$ follows because $|\mathbf{g}^{i|j}|$ and $c^{i|j}$ are symmetric in (i, j) . Finally, if $i \sim j$ and $j \sim k$, the triangle inequality $|\mathbf{g}^{i|k}| \leq |\mathbf{g}^{i|j}| + |\mathbf{g}^{j|k}|$ gives $\int_0^\infty |\mathbf{g}_t^{i|k}| dt < \infty$. By Itô’s formula,

$$\log \left(\frac{\kappa_t^i}{\kappa_t^k} \right) = \log \left(\frac{\kappa_0^i}{\kappa_0^k} \right) + \int_0^t \mathbf{g}_u^{i|k} du + \langle \mathbf{e}_i - \mathbf{e}_k, M_t \rangle.$$

Then, $\langle \mathbf{e}_i - \mathbf{e}_k, M \rangle = \log(\kappa^i/\kappa^k) - \log(\kappa_0^i/\kappa_0^k) - \int_0^\cdot |\mathbf{g}_t^{i|k}| dt$ is a local martingale. We have $\{i \sim j\} \cap \{j \sim k\} \subseteq \{\lim_{t \rightarrow \infty} \log(\kappa_t^i/\kappa_t^k) \text{ exists}\}$ from (16), hence $\langle \mathbf{e}_i - \mathbf{e}_k, M \rangle$

has a finite limit at infinity on $\{i \sim j\} \cap \{j \sim k\}$, which means that its quadratic variation up to infinity has to be finite on the latter event, i.e., $\int_0^\infty |c_t^{i|k}| dt < \infty$ on $\{i \sim j\} \cap \{j \sim k\}$, and the claim is proved. The same holds for the relationship \sim described in Remark 5.3 above for portfolios.

On the event $\{\kappa_\infty := \lim_{t \rightarrow \infty} \kappa_t \text{ exists}\} \cap \{i \sim j\}$ we have $\kappa_\infty^i = 0 \Leftrightarrow \kappa_\infty^j = 0$, and thus also $\kappa_\infty^i > 0 \Leftrightarrow \kappa_\infty^j > 0$; this is trivial in view of (16). A somewhat surprising partial converse to this last observation is given now.

Lemma 5.5 *For any pair (i, j) , we have $\Omega_b \cap \{\kappa_\infty^i > 0, \kappa_\infty^j > 0\} \subseteq \{i \sim j\}$.*

Proof Since V^κ / V^ρ has a strictly positive limit at infinity on Ω_b , we get that the local martingale V^{e^i} / V^ρ has a strictly positive limit at infinity on $\Omega_b \cap \{\kappa_\infty^i > 0\}$. According to Lemma A.2, this means that $\int_0^\infty |\mathbf{g}_t^{i|\rho}| dt = \int_0^\infty |\mathbf{g}_t^{\rho|i}| dt < \infty$. Then, on $\Omega_b \cap \{\kappa_\infty^i > 0, \kappa_\infty^j > 0\}$ we have both $\int_0^\infty |\mathbf{g}_t^{i|\rho}| dt < \infty$ and $\int_0^\infty |\mathbf{g}_t^{j|\rho}| dt < \infty$; since $|\mathbf{g}^{i|j}| \leq |\mathbf{g}^{i|\rho}| + |\mathbf{g}^{j|\rho}|$, we get $\int_0^\infty |\mathbf{g}_t^{i|j}| dt < \infty$. Now, the fact that $\lim_{t \rightarrow \infty} \log(\kappa_t^i / \kappa_t^j)$ exists on $\{\kappa_\infty^i > 0, \kappa_\infty^j > 0\}$ allows us to proceed as in Remark 5.4 and show that $\int_0^\infty c_t^{i|j} dt < \infty$. We conclude that $i \sim j$ on $\Omega_b \cap \{\kappa_\infty^i > 0, \kappa_\infty^j > 0\}$. \square

5.3 One company takes all

Now comes the main result of this section.

Theorem 5.6 $\Omega_b \cap \Sigma \subseteq \{\kappa_\infty \in \{\mathbf{e}_1, \dots, \mathbf{e}_d\}\}$. In particular, in a balanced and segregated market, κ_∞ exists \mathbb{P} -a.s. and is equal to a unit vector.

Proof This is a simple corollary of Lemma 5.5: On Ω_b , if we had $\kappa_\infty^i > 0$ and $\kappa_\infty^j > 0$ for any two companies i and j , we should have $i \sim j$; but the segregation event Σ is exactly the one where $i \not\sim j$ for all pairs of companies (i, j) . \square

Remark 5.7 This is a follow-up to the discussion in paragraph 4.2.4 on Equivalent Martingale Measures. Existence of an EMM \mathbb{Q} , coupled with Theorem 5.6, imply that for each $i \in \{1, \dots, d\}$ we have $\mathbb{Q}[\kappa_\infty^i = 1 | \mathcal{F}_0] = \kappa_0^i > 0$, thus $\mathbb{P}[\kappa_\infty^i = 1 | \mathcal{F}_0] > 0$ as well. This ceases to be true anymore if we consider balanced markets. Indeed, in the next section one finds an example of a balanced and segregated market, such that a specific company takes over the whole market with probability one.

6 Examples

We consider here a parametric “toy” market model in order to illustrate the results of the previous subsections and to clarify some points discussed. The market will consist of two companies, and their capitalizations are S^0 and S^1 . Under \mathbb{P} , $S^0 \equiv 1$, while $S_0^1 = 1$ and $dS_t^1 = S_t^1(a_t dt + \sigma_t dW_t)$, where a and σ are predictable processes, σ is strictly positive, and W is a one-dimensional Brownian motion. In the

three cases we consider below we always have $0 \leq a/\sigma^2 \leq 1/2$; it then turns out that $\rho = (1 - a/\sigma^2, a/\sigma^2)$ and easy computations show that

$$L_\infty = \frac{1}{2} \int_0^\infty \left| \frac{a_t}{\sigma_t} - \sigma_t \kappa_t^1 \right|^2 dt, \text{ as well as } \{0 \approx 1\} = \left\{ \frac{1}{2} \int_0^\infty |\sigma_t|^2 dt = \infty \right\}. \quad (17)$$

6.1 Case $a = 0$

This market is balanced. Indeed, $L_\infty = (1/2) \int_0^\infty |\sigma_t \kappa_t^1|^2 dt \leq (1/2) \int_0^\infty |\sigma_t S_t^1|^2 dt$; observe that $\int_0^\infty |\sigma_t S_t^1|^2 dt$ is the quadratic variation of the local martingale S^1 up to infinity, which should be finite, since S^1 has a limit at infinity. It follows that $\Omega_b = \{L_\infty < \infty\} = \Omega$.

Observe also that $\{0 \approx 1\} = \{\lim_{t \rightarrow \infty} S_t^1 = 0\} = \{\kappa_\infty = e_0\}$. Here, the limit in the event $\Omega_b \cap \{0 \approx 1\} = \{0 \approx 1\}$ is identified as being e_0 , and one sees that on $\{0 \approx 1\}$ we have $0 < \kappa_\infty^0 < 1$ as well as $0 < \kappa_\infty^1 < 1$. In a balanced market with equivalent companies the limiting capital distribution might not be trivial.

Assume now that $\mathbb{P}[\int_0^\infty |\sigma_t|^2 dt = \infty] = 1$; easy examples of this is when σ is a positive constant, or when S^1 is the inverse of a three-dimensional Bessel process. From the discussion above, the market is balanced and segregated. We note that there *cannot* exist any probability measure $\mathbb{Q} \sim \mathbb{P}$ such that κ is a \mathbb{Q} -martingale; for if there existed one, the bounded martingale κ^1 would be uniformly integrable, so that $0 = \mathbb{E}^\mathbb{Q}[\kappa_\infty^1] = \kappa_0^1 = 1/2$ should hold, which is impossible.

This example clearly shows that balanced markets form a strictly larger class than the ones satisfying the EMM hypothesis discussed in 4.2.4.

6.2 Case $\epsilon \leq a/\sigma^2 \leq 1/2 - \epsilon$

Here we assume the previous inequality holds for all $(\omega, t) \in \Omega \times \mathbb{R}_+$ for some $0 < \epsilon < 1/4$; for example, one can just pick some predictable, strictly positive process σ and then set $a = \sigma^2/4$.

As in the previous case $a = 0$, we have $\{0 \approx 1\} = \{\lim_{t \rightarrow \infty} S_t^1 = 0\} = \{\kappa_\infty^1 = 0\}$ —this follows from (17); just divide the equality

$$\log S_t^1 = \int_0^t \left(a_u - \frac{1}{2} \sigma_u^2 \right) du + \int_0^t \sigma_u dW_u$$

by $\int_0^t |\sigma_u|^2 dt$ and then use $a - \sigma^2/2 \leq -\epsilon\sigma^2$ as t tends to infinity. Because of this last fact, using $\epsilon \leq a/\sigma^2$ and (17) again, we easily get $\{0 \approx 1\} \subseteq \{L_\infty = \infty\} = \Omega_u = \Omega \setminus \Omega_b$. This example shows that the limiting capital distribution can be concentrated in one company even in the set where then market is not balanced.

6.3 Case $a = \sigma^2/2$

In this case, $\log S^1$ is a local martingale with quadratic variation $\int_0^{\cdot} |\sigma_t|^2 dt$, and thus on $\{0 \approx 1\} = \{\int_0^{\infty} |\sigma_t|^2 dt = \infty\}$ we have $\liminf_{t \rightarrow \infty} \kappa_t^1 = 0$ and $\limsup_{t \rightarrow \infty} \kappa_t^1 = 1$; obviously, the same relationships hold for κ^0 as well. On the other hand, on $\{\int_0^{\infty} |\sigma_t|^2 dt < \infty\}$ we have that $\lim_{t \rightarrow \infty} \kappa_t$ exists, and since $2L_{\infty} = \int_0^{\infty} |\sigma_t|(\kappa_t^1 - 1/2)^2 dt$ by (17), we have $L_{\infty} < \infty$. We conclude that

$$\Omega_u = \left\{ \int_0^{\infty} |\sigma_t|^2 dt = \infty \right\} = \left\{ \liminf_{t \rightarrow \infty} \kappa_t^i = 0, \limsup_{t \rightarrow \infty} \kappa_t^i = 1, \quad \text{for both } i = 0, 1 \right\},$$

which shows that the result of Proposition 5.1 cannot be strengthened. It also shows that it is exactly the unbalanced markets that bring diversity into the picture and the hope that not all capital will concentrate in one company only.

7 The quasi-left-continuous case

We now discuss all the previous results in a more general setting, where we allow for the processes of company capitalizations to have jumps. For notions regarding semimartingale theory used in the sequel, one can consult [Jacod and Shiryaev \(2003\)](#). Numbered subsections correspond to previous numbered sections, i.e., Subsect. 7.1 to Sect. 1, Subsect. 7.2 to Sect. 2, and so on.

7.1 The set-up

We denote by S^i the capitalization of company i . Each S^i , $i = 1, \dots, d$ is modeled as a semimartingale living on an underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to the filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ that satisfies the usual conditions. One extra ingredient that *has to* be added (in view of Example 7.2 later) is to allow for the capitalizations to become zero, which can be considered as *death*, or *annihilation* of the company. We define the *lifetime* of company i as $\zeta^i := \inf\{t \in \mathbb{R}_+ \mid S_{t-}^i = 0 \text{ or } S_t^i = 0\}$; each ζ^i is an \mathbf{F} -stopping time. After dying, companies cannot revive; thus we insist that $S_t^i \equiv 0$, for all $t \geq \zeta^i$. Note that—even though individual companies might die—we suppose the whole market lives forever; $\max_{1 \leq i \leq d} \zeta^i = +\infty$, \mathbb{P} -a.s.

We want to write an expression like:

$$dS_t^i = S_{t-}^i dX_t^i, \quad \text{for } t \in [0, \zeta^i], \quad i = 1, \dots, d. \quad (18)$$

where dX_t^i plays the equivalent rôle of $a_t^i dt + dM_t^i$ of (1). Let us assume for the moment that $\zeta^i = \infty$ for all $i = 1, \dots, d$, so that X^i can be defined as the *stochastic logarithm* of S^i : $X^i := \int_0^{\cdot} (dS_t^i / S_{t-}^i)$. Then, we know that if we fix the *canonical truncation function* $x \mapsto x\mathbb{I}_{\{|x| \leq 1\}}$ (\mathbb{I}_A will denote the indicator of a set A), the *canonical*

decomposition of the semimartingale $X = (X^1, \dots, X^d)$ is

$$X = B + M + [x \mathbb{I}_{\{|x| \leq 1\}}] * (\mu - \eta) + [x \mathbb{I}_{\{|x| > 1\}}] * \mu. \quad (19)$$

In the decomposition (19), B is predictable and of finite variation; M is a continuous local martingale; μ is the *jump measure* of X , i.e., the random measure on $\mathbb{R}_+ \times \mathbb{R}^d$ defined by $\mu([0, t] \times A) := \sum_{0 \leq s \leq t} \mathbb{I}_{A \setminus \{0\}}(\Delta X_s)$, for $t \in \mathbb{R}_+$ and $A \subseteq \mathbb{R}^d$; the asterisk “ $*$ ” denotes integration with respect to random measures; η is the *predictable compensator* of μ —it satisfies $[|x|^2 \wedge 1] * \eta_t < \infty$ for all $t \in \mathbb{R}_+$, and $\eta[\mathbb{R}_+ \times (-\infty, -1)^d] = 0$, since each S^i ($i = 1, \dots, d$) is constrained to be positive.

Since we do not know a-priori that $\zeta^i = \infty$ for all $i = 1, \dots, d$, we take the opposite direction of assuming the representation (19), and pick as *inputs* a continuous local martingale M , a quasi-left-continuous *semimartingale jump measure* μ , and a continuous process B that is locally of finite variation before a possible explosion to $-\infty$. The continuous local martingale M being obvious, we remark on the last two objects.

A *semimartingale jump measure* μ is a random counting measure on $\mathbb{R}_+ \times \mathbb{R}^d$ with $\mu(\mathbb{R}_+ \times \{0\}) = 0$ and $\mu(\{t\} \times \mathbb{R}^d)$ being $\{0, 1\}$ -valued for all $t \in \mathbb{R}_+$, such that its predictable compensator η exists and satisfies $[|x|^2 \wedge 1] * \eta_t < \infty$ for all $t \in \mathbb{R}_+$. μ being quasi-left-continuous means $\mu(\{\tau\} \times \mathbb{R}^d) = 0$ for all *predictable* stopping times τ ; this is equivalent to $\eta(\{t\} \times \mathbb{R}^d) = 0$ for all $t \in \mathbb{R}_+$. In other words, jumps are permitted as long as they only come in a totally unpredictable (inaccessible) way. It will also be assumed that $\mu[\mathbb{R}_+ \times (-\infty, -1)^d] = 0$ (equivalently, $\eta[\mathbb{R}_+ \times (-\infty, -1)^d] = 0$) to keep the company-capitalization processes positive.

The twist comes for the predictable finite-variation process B , for which we shall assume that its coefficient-processes can explode to $-\infty$ in finite time. In other words, for each $i = 1, \dots, d$ there exists a *strictly increasing* sequence of stopping times $(\zeta_n^i)_{n \in \mathbb{N}}$ such that the stopped process $(B_{\zeta_n^i \wedge t}^i)_{t \in \mathbb{R}_+}$ is continuous (thus predictable) and of finite variation, and that $\lim_{n \rightarrow \infty} B_{\zeta_n^i}^i = -\infty$. It is clear that we can choose $\zeta_n^i := \inf\{t \in \mathbb{R}_+ | B_t^i = -n\}$. We further define

$$\zeta^i := \left(\lim_{n \rightarrow \infty} \uparrow \zeta_n^i \right) \wedge \inf\{t \in \mathbb{R}_+ | \mu(t, -1) = 1\}. \quad (20)$$

where we write $\mu(t, -1)$ as short for $\mu(\{(t, -1)\})$. The last definition should be intuitively obvious: annihilation of company i happens either (a) when B^i explodes to $-\infty$ in which case we have a continuous transition of S^i to zero in that $S_{\zeta^i}^i = 0$, or (b) the first time when $\mu(t, -1) = 1$, where we have a jump down to zero; in this case we have $S_{\zeta^i}^i > 0$ and $S_{\zeta^i}^i = 0$.

Having these ingredients we now *define* the process X via (19), where we tacitly assume that $X^i = -\infty$ on $\llbracket \zeta^i, \infty \rrbracket$. We also define the company capitalizations S^i for $i = 1, \dots, d$ via (18), where we set $S^i = 0$ on $\llbracket \zeta^i, \infty \rrbracket$.

Setting $C := [M, M]$ to be the *quadratic covariation* process of M , the triple (B, C, η) is called the *triplet of predictable characteristics* of X . One can find a continuous, one-dimensional, strictly increasing process G such that the processes C and

η are absolutely continuous with respect to it, in the sense of the Eqs. (21) below—for instance, one can choose $G = \sum_{i=1}^d [M^i, M^i] + [|x|^2 \wedge 1] * \eta$. We shall also assume that each B^i , $i = 1, \dots, d$ is absolutely continuous with respect to G on the stochastic interval $[[0, \zeta^i]]$ —otherwise it can be shown that there are trivial opportunities for free lunches of the most egregious kind—one can check Karatzas and Kardaras (2007), Section 5, for more information. It then follows that we can write

$$B = \int_0^{\cdot} b_t dG_t, \quad C = \int_0^{\cdot} c_t dG_t, \quad \text{and } \eta([0, t] \times E) = \int_0^t \left(\int_{\mathbb{R}^d} \mathbb{I}_E(x) \nu_u(dx) \right) dG_u \quad (21)$$

for any Borel subset E of \mathbb{R}^d . Here, all b , c and ν are predictable, b is a vector process, c is a positive-definite matrix-valued process and ν is a process with values in the space of measures on \mathbb{R}^d that satisfy $\nu(\{0\}) = 0$ and integrate $x \mapsto 1 \wedge |x|^2$ (so-called *Lévy measures*). Each process b^i for $i = 1, \dots, d$ is G -integrable on each stochastic interval $[[0, \zeta_n^i]]$, but on the event $\{\zeta^i < \infty, S_{\zeta^i-} = 0\}$, b^i it is not integrable on $[[0, \zeta^i]]$.

The differential “ dG_t ” will be playing the rôle that “ dt ” was playing before—for example, an *interest rate process* now is a one-dimensional predictable process r such that $\int_0^t |r_u| dG_u < \infty$ for all $t \in \mathbb{R}_+$.

7.2 Perfectly balanced markets

The notion of a perfectly balanced market is exactly the same as before: we ask that κ is a vector (\mathbb{P}, \mathbf{F}) -martingale.

The first order of business is to find necessary and sufficient conditions in terms of the triplet (b, c, ν) for the market to be perfectly balanced. Itô’s formula gives that the drift part of the stochastic logarithm process $\int_0^{\cdot} (d\kappa_t^i / \kappa_{t-}^i)$ on $[[0, \zeta^i]]$ is

$$\int_0^{\cdot} \left(\langle \mathbf{e}_i - \kappa_{t-}, b_t - c_t \kappa_{t-} \rangle + \int_{\mathbb{R}^d} \left[\frac{\langle \mathbf{e}_i - \kappa_{t-}, x \rangle}{1 + \langle \kappa_{t-}, x \rangle} - \langle \mathbf{e}_i - \kappa_{t-}, x \rangle \mathbb{I}_{\{|x| \leq 1\}} \right] \nu_t(dx) \right) dG_t.$$

In a perfectly balanced market, this last quantity has to vanish—using same arguments as in the proof of Proposition 2.3 we get the following result.

Proposition 7.1 *The market is perfectly balanced if and only if there exists an interest rate process r such that the following relationship holds for each coordinate $i = 1, \dots, d$ on $[[0, \zeta^i]]$:*

$$b - c\kappa_- + \int \left[\frac{x}{1 + \langle \kappa_-, x \rangle} - x \mathbb{I}_{\{|x| \leq 1\}} \right] \nu(dx) = r\mathbf{1}. \quad (22)$$

Using (22) above one computes that in a perfectly balanced market the relative company capitalization κ^i for each $i = 1, \dots, d$ satisfies

$$\kappa^i = \kappa_0^i \mathcal{E} \left(\int_0^{\cdot} \langle e_i - \kappa_{t-}, dM_t \rangle + \left[\frac{\langle e_i - \kappa_-, x \rangle}{1 + \langle \kappa_-, x \rangle} \right] * (\mu - \eta) \right), \text{ on } \llbracket 0, \zeta^i \rrbracket \quad (23)$$

where \mathcal{E} is the stochastic exponential operator.

In order to get a result about existence of perfectly balanced markets similar to Theorem 2.5 one has to start with the continuous local martingale M and a quasi-left-continuous semimartingale jump measure μ and show that Eqs. (23) have a strong solution. Below, we show by example that even if we start with an initial distribution of capital κ_0 in the *open* simplex Δ^{d-1} (so that $\kappa_0^i > 0$ for all $i = 1, \dots, d$) and jumps of size -1 are not allowed by the jump measure, annihilation of a company might come at *finite* time—stock-killing times were not included just for the sake of generality, but they come up naturally if possibly unbounded jumps above are allowed for the company-capitalization processes.

Example 7.2 Consider a simple market with two companies (we call them 0 and 1) for which $\kappa_0^0 = \kappa_0^1 = 1/2$, $M \equiv 0$, and μ is a jump measure with at most one jump at time τ that is an exponential random variable, and size $l(\tau)$ for a deterministic function l given by $l(t) = (1 - e^{t/2}/2)^{-1} \mathbb{I}_{[0, 2 \log 2)}(t)$. Observe that there is no jump on $\{\tau > 2 \log 2\}$, an event of positive probability, and that $v_t(dx) = \mathbb{I}_{(0, \tau]} \delta_{(0, l_t)}(dx)$, where δ is the Dirac measure.

Now, according to (23) the process κ^1 should satisfy

$$\frac{d\kappa_t^1}{dt} = -\frac{\kappa_t^1(1 - \kappa_t^1)l_t}{1 + \kappa_t^1 l_t}, \text{ for all } t < \tau.$$

It can be readily checked that the solution of the previous (ordinary) differential equation for $t < \min\{\tau, 2 \log 2\}$ is $\kappa^1 = 1/l$. Thus, on $\{\tau \geq 2 \log 2\}$ (which has positive probability), we have $\kappa^1 = 0$ for all $t \geq 2 \log 2$, i.e., $\mathbb{P}[\zeta^1 < \infty] > 0$.

Theorem 7.3 Consider a continuous (\mathbb{P}, \mathbf{F}) -local martingale M and a quasi-left-continuous semimartingale jump measure μ . Then, for any \mathcal{F}_0 -measurable initial condition $\kappa_0 \equiv (\kappa_0^i)_{1 \leq i \leq d}$ with $\mathbb{P}[\kappa_0 \in \overline{\Delta}^{d-1}] = 1$ the stochastic differential equations (23) have a unique strong solution on $[0, \infty)$ that lives on $\overline{\Delta}^{d-1}$.

Select any interest-rate process r and any \mathcal{F}_0 -measurable initial random vector $S_0 = (S_0^i)_{1 \leq i \leq d}$ such that $S_0^i / \langle S_0, \mathbf{1} \rangle = \kappa_0^i$. For all $i = 1, \dots, d$, define ζ^i by (20) and also define b^i by (22) on the interval $\llbracket 0, \zeta^i \rrbracket$. With $B = \int_0^{\cdot} b_t dG_t$, if we define X via (19), then S as defined by (18) is a model of a perfectly balanced market.

Proof More or less, one follows the steps of the proof of Theorem 2.5, with some twists. We assume that the initial condition κ_0 lives on Δ^{d-1} —any company $i = 1, \dots, d$ for which $\kappa_0^i = 0$ can be safely disregarded, since then $\kappa^i \equiv 0$.

Set $K_n := [n^{-1}, 1 - n^{-1}]^d$ for all $n \in \mathbb{N}$; the coefficients of (23) are Lipschitz on K_n . A theorem on strong solutions of stochastic differential equations involving

random measures has to be invoked—one can check for example [Bichteler \(2002\)](#) (Proposition 5.2.25, page 297) for existence of solutions of equations of the form (23) in the case of Lipschitz coefficients. We infer the existence of an increasing sequence of stopping times $(\tau_n)_{n \in \mathbb{N}}$ such that $\kappa_t \in K_n$ for all $t < \tau_n$ and $\kappa_{\tau_n} \in \mathbb{R}^d \setminus K_n$. Using (23) one can show that $\langle \kappa, \mathbf{1} \rangle$ is constant on $[[0, \tau_n]]$ — since $\kappa_0 \in \Delta^{d-1}$ we have $\kappa_t \in \Delta^{d-1}$ for all $t < \tau_n$. Now, we claim that $\kappa_{\tau_n} \in \overline{\Delta}^{d-1}$. Since $\langle \kappa_{\tau_n}, \mathbf{1} \rangle = 1$ we only need show that $\kappa_{\tau_n}^i \geq 0$ for all $i = 1, \dots, d$. If $\mu(\{\tau_n\} \times \mathbb{R}^d) = 0$ this is trivial. Otherwise, let $\xi_n \in [-1, \infty)^d$ the (random) point such that $\mu(\tau_n, \xi_n) = 1$; (23) gives

$$\kappa_{\tau_n}^i = \kappa_{\tau_n-}^i \left(\frac{1 + \xi_n^i}{1 + \langle \kappa_{\tau_n-}, \xi_n \rangle} \right) \geq 0,$$

since $\xi_n^i \geq -1$ and $\langle \kappa_{\tau_n-}, \xi_n \rangle > -1$ in view of the fact that $\kappa_{\tau_n-} \in \Delta^{d-1}$.

Pasting solutions together we get that there exists a stopping time τ such that $\kappa_t \in \Delta^{d-1}$ for all $t < \tau$ and $\kappa_\tau \in \partial \Delta^{d-1}$ on $\{\tau < \infty\}$. Unlike the proof of Theorem 2.5 we cannot hope now that $\mathbb{P}[\tau < \infty] = 0$, as Example 7.2 above shows. Rather, we set $\zeta^i = \tau$ if $\kappa_\tau^i = 0$ for $i = 1, \dots, d$.

We have constructed a solution to (23) on the stochastic interval $[[0, \tau]]$. On the event $\{\tau < \infty\}$ we continue the construction of the solution to (23) inductively, removing all companies that have died. In at most $d - 1$ steps we either have constructed the solution for all $t \in \mathbb{R}^d$, or only one company (say, i) has remained in which case we shall have $\kappa = \mathbf{e}_i$ from then onwards.

7.3 Perfect balance and growth

Growth-optimality of a portfolio and the market are now defined, and their relation to perfect balance is established.

A portfolio is a d -dimensional predictable and X -integrable processes, and from now onwards we restrict attention to the \mathfrak{C} -constrained class $\Pi_{\mathfrak{C}}$ where $\mathfrak{C} \equiv \overline{\Delta}^{d-1}$. If V^π denotes the wealth process generated by π we have

$$\frac{dV_t^\pi}{V_{t-}^\pi} = \sum_{i=1}^d \pi_t^i \frac{dS_t^i}{S_{t-}^i} + \left(1 - \sum_{i=1}^d \pi_t^i \right) r_t dG_t,$$

where r is some interest rate process coming from a bank in the market.

The market portfolio is not κ now, but rather its left-continuous version κ_- (the vector process κ as appears in (2) is not in general predictable, but only adapted and right-continuous). It is trivial to check that $V^{\kappa_-} = \langle S, \mathbf{1} \rangle / \langle S_0, \mathbf{1} \rangle$.

The concept of growth of a portfolio is sometimes not well-defined, as the log-wealth process $\log V^\pi$ might not be a special semimartingale, which means that its finite-variation part fails to exist. In order to define a growth optimal portfolio ρ , we use the idea contained in Remark 3.2: we ask that V^π / V^ρ is a supermartingale for all $\pi \in \Pi_{\mathfrak{C}}$. It turns out (one can check [Karatzas and Kardaras 2007](#), for example) that

this requirement is equivalent to $\text{rel}(\pi|\rho) \leq 0$ for all $\pi \in \Pi_{\mathcal{C}}$, where the *relative rate of return* process is

$$\begin{aligned} \text{rel}(\pi|\rho) := & \langle \pi - \rho, b - r\mathbf{1} \rangle - \langle \pi - \rho, c\rho \rangle \\ & + \int \left[\frac{\langle \pi - \rho, x \rangle}{1 + \langle \rho, x \rangle} - \langle \pi - \rho, x \rangle \mathbb{I}_{\{|x| \leq 1\}} \right] \nu(dx). \end{aligned} \quad (24)$$

The market will be called a growth market if κ_- is growth-optimal according to this last definition. It is easily shown that in order to have a growth market we must have (22) holding, where r is now the banking interest rate.

Exactly the same remarks on interest rates hold as the ones in Subsect. 3.3—the concepts of perfect balance and growth in markets are thus equivalent.

7.4 Balanced markets

To define the loss-of-balance process, let ρ be the growth-optimal portfolio in the class $\Pi_{\mathcal{C}}$ and set

$$L := \int_0^{\cdot} \left(-\text{rel}(\kappa_{t-}|\rho_t) + \frac{1}{2} c_t^{\kappa_-|\rho} \right) dG_t + \left[1 \wedge \left| \log \frac{1 + \langle \kappa_-, x \rangle}{1 + \langle \rho, x \rangle} \right|^2 \right] * \eta,$$

where $c^{\pi_1|\pi_2} := \langle \pi_2 - \pi_1, c(\pi_2 - \pi_1) \rangle$ for two portfolios π_1 and π_2 . As before, set $\Omega_b := \{L_{\infty} < \infty\}$ and $\Omega_u := \Omega \setminus \Omega_b = \{L_{\infty} = \infty\}$. The above definition of L is slightly different than the one of Definition 4.1 for the case of Itô processes, but for this special case it is easy to see that the sets Ω_b and Ω_u that are obtained using the two definitions are the same—and this is the only thing of importance.

With a little help from Lemma A.4 (more precisely, the generalization of its result as discussed in Remark A.5) we get that $\Omega_b = \{\lim_{t \rightarrow \infty} (V_t^{\kappa_-} / V_t^{\rho}) > 0\}$ and $\Omega_u = \{\lim_{t \rightarrow \infty} (V_t^{\kappa_-} / V_t^{\rho}) = 0\}$. Based on this characterization of the event of balanced outcomes, Theorem 4.4 can be proved for our more general case now.

7.5 Limiting capital distribution of balanced markets

Of course, the event-inclusion $\Omega_b \subseteq \{\kappa_{\infty} := \lim_{t \rightarrow \infty} \kappa_t \text{ exists}\}$ follows exactly from the equivalent of Theorem 4.4 in the quasi-left-continuous case—a limiting capital distribution exists for the balanced outcomes.

Two companies are equivalent (we write $i \sim_{\omega} j$) if $d^{i|j}(\omega) = \infty$, where

$$d^{i|j} := \int_0^{\infty} \left(|\text{rel}(e_i|\rho_t) - \text{rel}(e_j|\rho_t)| + \frac{1}{2} c_t^{i|j} \right) dG_t + \left[1 \wedge \left| \log \frac{1 + x^i}{1 + x^j} \right|^2 \right] * \eta_{\infty} \quad (25)$$

is a measure of distance between two companies in an ω -by- ω basis. Again, this definition does not fully agree with the one given in (15), but it is easy to see that the events

$\{i \sim j\}$ are identical under both definitions. Segregated markets and the segregation set Σ are formulated exactly as in Definition 5.2.

We again have $\Omega_b \cap \{\kappa_\infty^i > 0, \kappa_\infty^j > 0\} \subseteq \{i \sim j\}$. The proof follows the steps of Lemma 5.5, invoking Lemma A.4 (actually, Remark A.5) from the Appendix. Then, Theorem 5.6 follows trivially: on balanced outcomes that segregation of companies holds, one company will take all.

A Limiting behavior of local Martingales

The proof of the following result is well-known for continuous-path semimartingales—for the slightly more general case described below, the proof is the same.

Lemma A.1 *Let $X = M + x * (\mu - \eta)$ be a local martingale, where M is a continuous local martingale and μ is the jump measure of X with η its predictable compensator. We assume that X has bounded jumps: $|\Delta X| \leq c$ for some constant $c \geq 0$. Then, with $B := [M, M] + |x|^2 * \eta$ we have $\{\lim_{t \rightarrow \infty} X_t \text{ exists in } \mathbb{R}\} = \{B_\infty < +\infty\}$, while on the event $\{B_\infty = +\infty\}$ we have $\lim_{t \rightarrow \infty} (X_t / B_t) = 0$.*

This allows one to prove the following lemma.

Lemma A.2 *For a continuous local martingale M , consider the exponential local martingale $\mathcal{E}(M) = \exp(M - [M, M]/2)$. Then, $\mathcal{E}(M)_\infty := \lim_{t \rightarrow \infty} \mathcal{E}(M)_t$ exists and is \mathbb{R}_+ -valued. Further, $\{[M, M]_\infty < +\infty\} = \{\mathcal{E}(M)_\infty > 0\}$.*

Proof Existence of $\mathcal{E}(M)_\infty$ follows from the supermartingale convergence theorem. Lemma A.1 gives $\{[M, M]_\infty < +\infty\} = \{\lim_{t \rightarrow \infty} \mathcal{E}(M)_t \in \mathbb{R}\}$; thus $\{[M, M]_\infty < +\infty\} \subseteq \{\mathcal{E}(M)_\infty > 0\}$. For the other inclusion, Lemma A.1 again gives that on $\{[M, M]_\infty = +\infty\}$ we have $\lim_{t \rightarrow \infty} (\log \mathcal{E}(M)_t / [M, M]_t) = -1/2$; this means that $\lim_{t \rightarrow \infty} \log \mathcal{E}(M)_t = -\infty$, or $\mathcal{E}(M)_\infty = 0$ and we are done.

In order to prove the equivalent of Lemma A.2 for general semimartingales, a “strong law of large numbers” result for increasing processes will be needed.

Lemma A.3 *Let A be an increasing, right-continuous and adapted process with $|\Delta A| \leq c$ for some constant $c > 0$, and let \tilde{A} be its predictable compensator, so that $A - \tilde{A}$ is a local martingale. Then, we have $\{A_\infty < \infty\} = \{\tilde{A}_\infty < \infty\}$ and on $\{\tilde{A}_\infty = \infty\}$ we have $\lim_{t \rightarrow \infty} (A_t / \tilde{A}_t) = 1$.*

Proof It is easy to see that we can assume without loss of generality that A is pure-jump and quasi-left-continuous (if not, decompose A into a part as described and another part that is predictable; this second part can be subtracted from both A and \tilde{A}). Let η be the predictable compensator of the jump measure of A ; observe then that $\tilde{A} = x * \eta$ and if $N := A - \tilde{A}$, then $B := \widetilde{[N, N]} = |x|^2 * \eta$. Since A has jumps bounded by c , it is clear that $B \leq c\tilde{A}$.

On $\{\tilde{A}_\infty < +\infty\}$ we have $B_\infty < +\infty$, so that N_∞ exists, and thus $A_\infty < +\infty$.

Now, work on $\{\tilde{A}_\infty = +\infty\}$. If $B_\infty < +\infty$, M_∞ exists and is real-valued, so obviously $\lim_{t \rightarrow \infty} (A_t - \tilde{A}_t) / \tilde{A}_t = 0$. If $B_\infty = +\infty$, we have $\lim_{t \rightarrow \infty} (A_t - \tilde{A}_t) / B_t = 0$, so that also $\lim_{t \rightarrow \infty} (A_t - \tilde{A}_t) / \tilde{A}_t = 0$, and this completes the proof.

Lemma A.4 Let X and Y be local martingales with $\Delta X > -1$, $\Delta Y > -1$ (then, $\mathcal{E}(X)$ and $\mathcal{E}(Y)$ are positive local martingales). Write $X = M + x * (\mu - \eta)$ and $Y = N + y * (\mu - \eta)$ with M and N being continuous local martingales, μ the 2-dimensional jump measure of (X, Y) and η its predictable compensator. Then,

1. $\{\mathcal{E}(X)_\infty > 0\} = \{[M, M]_\infty/2 + [1 \wedge |\log(1+x)|^2] * \eta_\infty < +\infty\}.$
2. $\{\mathcal{E}(X)_\infty > 0, \mathcal{E}(Y)_\infty > 0\} \subseteq \{d^{X|Y} < +\infty\},$ where we have set

$$d^{X|Y} := \frac{1}{2}[M - N, M - N]_\infty + \left[1 \wedge \left| \log \left(\frac{1+x}{1+y} \right) \right|^2 \right] * \eta_\infty$$

Proof For (1), the definition of the stochastic exponential gives

$$\log \mathcal{E}(X) = X - \frac{1}{2}[M, M] - [x - \log(1+x)] * \mu.$$

Since $\Delta \log \mathcal{E}(X) = \log(1 + \Delta X)$, on $\{\mathcal{E}(X)_\infty > 0\} = \{\log \mathcal{E}(X)_\infty \in \mathbb{R}\}$ we should have $|\log(1 + \Delta X_t)| > 1$ for a finite (path-dependent) number of $t \in \mathbb{R}_+$ —equivalently, we must have that $\mathbb{I}_{\{|\log(1+x)|>1\}} * \mu_\infty < +\infty$ and then Lemma A.3 implies $\mathbb{I}_{\{|\log(1+x)|>1\}} * \eta_\infty < +\infty$. Now, if we subtract the semimartingale $[\log(1+x)]\mathbb{I}_{\{|\log(1+x)|>1\}} * \mu_\infty$ (which is actually only a finite sum) from $\log \mathcal{E}(X)$, what remains is a semimartingale with bounded (by one) jumps. The canonical representation of the semimartingale $\log \mathcal{E}(X) - [\log(1+x)]\mathbb{I}_{\{|\log(1+x)|>1\}} * \mu$ into a sum of a predictable finite-variation part (first two terms in (26) below) and a local martingale part (last two terms):

$$-\frac{1}{2}[M, M] + [x - \log(1+x)]\mathbb{I}_{\{|\log(1+x)|\leq 1\}} * \eta + M + [\log(1+x)]\mathbb{I}_{\{|\log(1+x)|\leq 1\}} * (\mu - \eta). \quad (26)$$

This last semimartingale must have a real limit at infinity. Observe that on $\{[M, M]_\infty + [|\log(1+x)|^2]\mathbb{I}_{\{|\log(1+x)|\leq 1\}} * \eta_\infty = +\infty\}$ this cannot happen, because Lemma A.1 would give that the limit at infinity of the ratio of (26) to its predictable finite variation part would be equal to 1, which would imply that the semimartingale (26) does not have a limit. This completes the proof of (1).

Let us proceed to (2); we work on $\{\mathcal{E}(X)_\infty > 0, \mathcal{E}(Y)_\infty > 0\}$. Part (1) of this lemma gives $[M - N, M - N]_\infty \leq 2[M, M]_\infty + 2[N, N]_\infty < +\infty$. Now, define

$$\Lambda := \left\{ (x, y) \in (-1, \infty)^2 \mid \left| \log \left(\frac{1+x}{1+y} \right) \right| \leq 1 \right\}$$

as well as $\Lambda_x := \{(x, y) \in (-1, \infty)^2 \mid |\log(1+x)| \leq 1/2\}$ and $\Lambda_y := \{(x, y) \in (-1, \infty)^2 \mid |\log(1+y)| \leq 1/2\}$. With the prime “ $'$ ” denoting the complement of a set, we have $\Lambda' \subseteq \Lambda'_x \cup \Lambda'_y$, so $\mathbb{I}_{\Lambda'} * \eta_\infty < +\infty$ as discussed before. We then only have to show that $[\mathbb{I}_\Lambda |\log((1+x)/(1+y))|^2] * \eta_\infty < +\infty$. Since we have that $[\mathbb{I}_{\Lambda_x} |\log(1+x)|^2] * \eta_\infty < +\infty$ and $[\mathbb{I}_{\Lambda_y} |\log(1+y)|^2] * \eta_\infty < +\infty$ holds from part

(1) of this lemma, we need only show that $|f|^2 * \eta_\infty < +\infty$, where

$$f(x, y) := \mathbb{I}_\Lambda \log \left(\frac{1+x}{1+y} \right) - \mathbb{I}_{\Lambda_x} \log(1+x) + \mathbb{I}_{\Lambda_y} \log(1+y).$$

It is clear from part (1) that $[\mathbb{I}_{\Lambda'} |f|^2] * \eta_\infty < +\infty$. Now, on $\Lambda \cap \Lambda_x \cap \Lambda_y$ we have $f = 0$, while on $\Lambda \cap \Lambda'_x \cap \Lambda'_y$ we have $|f| \leq 1$. For $(x, y) \in \Lambda \cap \Lambda_x \cap \Lambda'_y$ we have $f(x, y) = -\log(1+y)$, which (using the triangle inequality) cannot be more than $3/2$ in absolute value. The similar thing holds on $\Lambda \cap \Lambda'_x \cap \Lambda_y$, so finally $[\mathbb{I}_\Lambda |f|^2] * \eta_\infty \leq (3/2)[\mathbb{I}_{\Lambda \cap (\Lambda_x \cap \Lambda_y)'}] * \eta_\infty < +\infty$, which completes the proof.

Remark A.5 Lemma A.4 can be extended in the case where X and Y are of the form $X = -A + M + x * (\mu - \eta)$ and $Y = -B + N + y * (\mu - \eta)$, where A and B are *increasing* and *continuous* adapted processes. In that case we have

1. $\{\mathcal{E}(X)_\infty > 0\} = \{A_\infty + [M, M]_\infty/2 + [1 \wedge |\log(1+x)|^2] * \eta_\infty < +\infty\}$.
2. $\{\mathcal{E}(X)_\infty > 0, \mathcal{E}(Y)_\infty > 0\} \subseteq \{d^{X|Y} < +\infty\}$, where

$$d^{X|Y} := \int_0^\infty d|A - B|_t + \frac{1}{2}[M - N, M - N]_\infty + \left[1 \wedge \left| \log \left(\frac{1+x}{1+y} \right) \right|^2 \right] * \eta_\infty$$

We can extend the discussion further when A or B might explode to ∞ in finite time, i.e., if the lifetimes $\zeta^X := \inf\{t \in \mathbb{R}_+ | X_t = -\infty\}$ and $\zeta^Y := \inf\{t \in \mathbb{R}_+ | Y_t = -\infty\}$ are finite, exactly as described in Subsect. 7.1 of the main text.

References

Bichteler, K.: Stochastic Integration with Jumps. Cambridge: Cambridge University Press 2002

Fama, E.F.: Efficient capital markets: a review of theory and empirical work. *J Financ* (1970)

Fama, E.F., MacBeth, J.D.: Long-term growth in a short-term market. *J Financ* **29**, 857–885 (1974)

Fernholz, E.R.: Stochastic Portfolio Theory. New York: Springer (2002)

Fernholz, R., Karatzas, I., Kardaras, C.: Diversity and relative arbitrage in equity markets. *Financ Stoc* **9**, 1–27 (2005)

Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. New York: Springer (2003)

Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. *Financ Stoc* (2007) (to appear)

Karatzas, I., Shreve, S.E.: Brownian Motion and Stochastic Calculus. New York: Springer (1998)

Karatzas, I., Shreve, S.E.: Methods of Mathematical Finance. New York: Springer (1998)

Roll, R.: Evidence on the ‘growth optimum’ model. *J Financ* **28**, 551–566 (1973)

Yan, J.A.: A new look at the fundamental theorem of asset pricing. *J Korean Math Soc* **35**(3), 659–673 (1998)