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Diversity and relative arbitrage in equity markets

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Abstract. An equity market is called "diverse" if no single stock is ever allowed to dominate the entire market in terms of relative capitalization. In the context of the standard Itô-process model initiated by Samuelson (1965) we formulate this property (and the allied, successively weaker notions of "weak diversity" and "asymptotic weak diversity") in precise terms. We show that diversity is possible to achieve, but delicate. Several examples are provided which illustrate these notions and show that weakly-diverse markets contain relative arbitrage opportunities: it is possible to outperform or underperform such markets over any given time-horizon. The existence of this type of relative arbitrage does not interfere with the development of contingent claim valuation, and has consequences for the pricing of long-term warrants and for put-call parity. Several open questions are suggested for further study.

Key words: Financial markets, portfolios, diversity, relative arbitrage, order statistics, local times, stochastic differential equations, strict local martingales

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1 Introduction

The descriptive notion of diversity for equity markets was introduced and studied recently by Fernholz (1999, 2002). It postulates, roughly, that no individual stock ever be allowed to dominate the entire market in terms of relative capitalization. In the context of the standard Itô-process, geometric-Brownian-Motion-based model introduced by Samuelson (1965), it is shown in Fernholz (2002) how to generate fully-invested, all-long portfolios that outperform a diverse market over sufficiently long time-horizons and how to exploit this property for passive asset management. The present paper complements this effort by showing that diversity is indeed possible under appropriate, though rather delicate, conditions. These mandate, roughly, that the largest stock have strongly negative rate of growth, resulting in a sufficiently strong drift away from an appropriate boundary, and that all other stocks have sufficiently high rates of growth. We also show that in diverse markets, relative arbitrage opportunities exist over arbitrary time-horizons: it is possible to construct two portfolios so that one outperforms the other with probability one. In particular, no equivalent martingale measure can exist for such markets.

Section 2 sets up the model and the notation used throughout the paper. Section 3 introduces the *market portfolio*, in terms of which the notion of *diversity* and the allied, successively weaker notions of *weak diversity* and *asymptotic weak diversity* are defined in Sect. 4. The dynamics for the ranked market weights are studied in Sect. 5, and in terms of them sufficient conditions for diversity are established in Sect. 6. These are illustrated by means of several examples, including models that are weakly diverse but fail to be diverse. Section 7 contains a model for which weak diversity fails on finite time-horizons but prevails as the time-horizon becomes infinite, in the asymptotic sense of Sect. 4.

We study in (4.4)–(4.5) a diversity-weighted portfolio that outperforms significantly any weakly-diverse market over sufficiently long time-horizons, leading to arbitrage relative to the market. In Sect. 8 we introduce the *mirror portfolios* and study their properties; these are then used to show that, in the context of a weakly-diverse market, it is possible to outperform (or underperform) the market-portfolio over arbitrary time-horizons.

Finally, in Sect. 9 we study diverse market models that contain a risk-free instrument and allow for general trading strategies (with short-selling and borrowing). Such models admit no equivalent martingale measure, no arbitrage opportunities in the "classical" sense of non-negative wealth with probability one and positive wealth with positive probability, but do admit so-called "free lunches with vanishing risk" (or "free snacks"). Nevertheless, familiar techniques for hedging contingent claims can be carried out in their context. This has ramifications for put-call parity and for the hedging prices of call-options over exceedingly long time-horizons: these hedging prices are shown to approach zero rather than the initial stock-value (as they do when an equivalent martingale measure exists for every finite time-horizon).

2 The model

We shall place ourselves in the standard Itô-process model for a financial market which goes back to Samuelson (1965). This model contains n risky assets (stocks), with values-per-share $X_i(\cdot)$ driven by m independent Brownian motions as follows:

$$dX_i(t) = X_i(t) \left[b_i(t)dt + \sum_{\nu=1}^{m} \sigma_{i\nu}(t) dW_\nu(t) \right], \quad i = 1, \dots, n \tag{2.1}$$

for $0 \le t < \infty$, with $m \ge n$. Here $X_i(t)$ stands for the value of the i^{th} asset at time t and $W(\cdot) = (W_1(\cdot), \dots, W_m(\cdot))'$ is a vector of m independent standard Brownian motions, the "factors" of the model. All processes are defined on a probability space $(\Omega, \mathcal{F}, P)$ and are adapted to a given filtration $\mathbf{F} = \{\mathcal{F}(t)\}_{0 \le t < \infty}$ with $\mathcal{F}(0) = \{\emptyset, \Omega\}$ mod. P; this satisfies the "usual conditions" (right-continuity, augmentation by $P-$negligible sets) and may be strictly larger than the one generated by the driving $m-$dimensional Brownian motion $W(\cdot)$.

The vector-valued process $b(\cdot) = \big(b_1(\cdot), \dots, b_n(\cdot)\big)'$ of *rates of return*, and the $(n \times m)$-matrix-valued process $\sigma(\cdot) = \big\{\sigma_{i\nu}(\cdot)\big\}_{1 \le i \le n, 1 \le \nu \le m}$ of *volatilities*, are assumed to be $\mathbf{F}-$progressively measurable and to satisfy almost surely (a.s.) the conditions

$$\int_0^T ||b(t)||^2 dt < \infty, \quad \forall \ T \in (0, \infty), \tag{2.2}$$

$$\varepsilon ||\xi||^2 \le \xi' \sigma(t) \sigma'(t) \xi \le M ||\xi||^2, \quad \forall \ t \in [0, \infty) \quad \text{and} \quad \xi \in I\!R^n \tag{2.3}$$

for some real constants $M > \varepsilon > 0$. We may re-write (2.1) in the equivalent form

$$d\Big(\log X_i(t)\Big) = \gamma_i(t)dt + \sum_{\nu=1}^{m} \sigma_{i\nu}(t) dW_\nu(t), \quad i = 1, \dots, n. \tag{2.4}$$

Here we have denoted by $\gamma_i(t) := b_i(t) - \frac{1}{2} a_{ii}(t)$, $i = 1, \dots, n$ the individual stock *growth-rates*, and by $a(\cdot) = \big\{a_{ij}(\cdot)\big\}_{1 \le i,j \le n}$ the $(n \times n)$-matrix of variation-covariation rate processes

$$a_{ij}(t) := \sum_{\nu=1}^{m} \sigma_{i\nu}(t) \sigma_{j\nu}(t) = \Big(\sigma(t)\sigma'(t)\Big)_{ij} = \frac{d}{dt} \langle \log X_i, \log X_j \rangle(t). \tag{2.5}$$

Placed in the above market-model $\mathcal{M}$ of (2.1)–(2.3), an economic agent can decide what proportion $\pi_i(t)$ of his wealth to invest in each of the stocks $i = 1, \dots, n$ at every time $t \in [0, \infty)$. The resulting *portfolio process* $\pi(\cdot) = \big(\pi_1(\cdot), \dots, \pi_n(\cdot)\big)'$ takes values in the set

$$\Delta_+^n = \left\{ (\pi_1, \dots, \pi_n) \in I\!R^n \Big| \pi_1 \ge 0, \dots, \pi_n \ge 0 \quad \text{and} \quad \sum_{i=1}^{n} \pi_i = 1 \right\}$$

(i.e., there is no money-market or hoarding of wealth) and is $\mathbf{F}-$progressively measurable. Starting with initial capital $z > 0$, the *value process* $Z^\pi(\cdot)$ of the portfolio $\pi(\cdot)$ satisfies

$$\frac{dZ^\pi(t)}{Z^\pi(t)} = \sum_{i=1}^n \pi_i(t) \cdot \frac{dX_i(t)}{X_i(t)} = b^\pi(t)dt + \sum_{\nu=1}^m \sigma_\nu^\pi(t)dW_\nu(t), \quad Z^\pi(0) = z \tag{2.6}$$

by analogy with Eq. (2.1), where

$$b^\pi(t) := \sum_{i=1}^n \pi_i(t)b_i(t), \qquad \sigma_\nu^\pi(t) := \sum_{i=1}^n \pi_i(t)\sigma_{i\nu}(t) \tag{2.7}$$

for $\nu = 1, \ldots, m$, are respectively the rate-of-return and the volatility coëfficients of the portfolio. As in (2.4) we may write the solution of the Eq. (2.6) in the form

$$\begin{aligned} d\Big(\log Z^\pi(t)\Big) &= \gamma^\pi(t)dt + \sum_{\nu=1}^m \sigma_\nu^\pi(t)dW_\nu(t), \qquad \text{with} \\ \gamma^\pi(t) &:= \sum_{i=1}^n \pi_i(t)\gamma_i(t) + \gamma_*^\pi(t), \\ \gamma_*^\pi(t) &:= \frac{1}{2}\left(\sum_{i=1}^n \pi_i(t)a_{ii}(t) - \sum_{i=1}^n\sum_{j=1}^n \pi_i(t)a_{ij}(t)\pi_j(t)\right) \end{aligned} \tag{2.8}$$

denoting, respectively, the *growth-rate* and the *excess-growth-rate* of the portfolio $\pi(\cdot)$.

In order to set the stage for notions and developments that follow, let us introduce the "order-statistics" notation for the weights

$$\max_{1\le i\le n} \pi_i(t) =: \pi_{(1)}(t) \ge \pi_{(2)}(t) \ge \ldots \ge \pi_{(n-1)}(t) \ge \pi_{(n)}(t) := \min_{1\le i\le n} \pi_i(t) \tag{2.9}$$

of a portfolio $\pi(\cdot)$, ranked at time t from the largest $\pi_{(1)}(t)$ to the smallest $\pi_{(n)}(t)$.

We introduce two notions of **relative arbitrage**. Given any two portfolios $\pi(\cdot)$, $\rho(\cdot)$ with initial capital $Z^\pi(0) = Z^\rho(0) = z > 0$, we shall say that $\pi(\cdot)$ *represents relative to* $\rho(\cdot)$

• *an arbitrage opportunity over the fixed, finite time-horizon* $[0, T]$ if we have

$$P\left[Z^\pi(T) \ge Z^\rho(T)\right] = 1 \quad \text{and} \quad P\left[Z^\pi(T) > Z^\rho(T)\right] > 0; \tag{2.10}$$

• *a superior long-term growth opportunity*, if

$$\mathcal{L}^{\pi,\rho} := \underline{\lim}_{T\to\infty} \frac{1}{T} \log\left(\frac{Z^\pi(T)}{Z^\rho(T)}\right) > 0 \quad \text{holds a.s.} \tag{2.11}$$

3 The market portfolio

Suppose we normalize so that each stock has one share outstanding; then the stock-value $X_i(t)$ can be interpreted as the capitalization of the i^{th} company at time t, and the quantities

$$Z(t) := X_1(t) + \ldots + X_n(t) \quad \text{and} \quad \mu_i(t) := \frac{X_i(t)}{Z(t)}, \quad i = 1, \ldots, n \tag{3.1}$$

as the total capitalization of the market and the relative capitalizations of the individual companies, respectively. Since $0 < \mu_i(t) < 1, \ \ \forall\ i = 1, \ldots, n$ and $\sum_{i=1}^n \mu_i(t) = 1$, we may think of the vector process $\mu(\cdot) = \big(\mu_1(\cdot), \ldots, \mu_n(\cdot)\big)'$ as a portfolio rule that invests a proportion $\mu_i(t)$ of current wealth in the i^{th} asset, at all times $t \in [0, \infty)$. Then the resulting value-process $Z^\mu(\cdot)$ satisfies

$$\frac{dZ^\mu(t)}{Z^\mu(t)} = \sum_{i=1}^n \mu_i(t) \cdot \frac{dX_i(t)}{X_i(t)} = \sum_{i=1}^n \frac{dX_i(t)}{Z(t)} = \frac{dZ(t)}{Z(t)},$$

as postulated by (2.6) and (3.1); and if we start with initial capital $Z^\mu(0) = Z(0)$ we get $Z^\mu(\cdot) \equiv Z(\cdot)$, the total market capitalization. In other words, investing according to the portfolio process $\mu(\cdot)$ amounts to ownership of the entire market in proportion to the original investment. For this reason we call $\mu(\cdot)$ the *market portfolio* for $\mathcal{M}$.

4 Notions of diversity

The notion of "diversity" for a financial market corresponds to the intuitive and descripive idea that no single company should be allowed to dominate the entire market in terms of relative capitalization. To make this precise, let us say that the model $\mathcal{M}$ of (2.1)–(2.3) is *diverse* on the time-horizon $[0, T]$, if there exists a number $\delta \in (0, 1)$ such that the quantities of (3.1) satisfy almost surely

$$\mu_{(1)}(t) < 1 - \delta, \quad \forall\ 0 \le t \le T \tag{4.1}$$

in the notation of (2.9). In a similar vein, we say that $\mathcal{M}$ is *weakly diverse* on the time-horizon $[0, T]$ if for some $\delta \in (0, 1)$ we have

$$\frac{1}{T} \int_0^T \mu_{(1)}(t) dt < 1 - \delta \tag{4.2}$$

almost surely. We say that $\mathcal{M}$ is *uniformly weakly diverse* over $[T_0, \infty)$, if there exists a $\delta \in (0, 1)$ such that (4.2) holds a.s. for every $T \in [T_0, \infty)$. And $\mathcal{M}$ is called *asymptotically weakly diverse* if, for some $\delta \in (0, 1)$, we have almost surely:

$$\overline{\lim}_{T\to\infty} \frac{1}{T} \int_0^T \mu_{(1)}(t) dt < 1 - \delta. \tag{4.3}$$

The first two of these notions were introduced in the paper by Fernholz (1999) and are studied in detail in the recent monograph Fernholz (2002). In particular, it is shown in Example 3.3.3 of this book that *if the model $\mathcal{M}$ of (2.1)–(2.3) is weakly diverse, then it contains arbitrage opportunities relative to the market portfolio.*

We provide here another example of such an arbitrage opportunity, in a weakly diverse market and for the so-called **diversity-weighted portfolio** $\pi^{(p)}(\cdot) = \big(\pi_1^{(p)}(\cdot), \dots, \pi_n^{(p)}(\cdot)\big)'$. For some fixed $0 < p < 1$, this is defined in terms of the market portfolio $\mu(\cdot)$ of (3.1), by

$$\pi_i^{(p)}(t) := \frac{\big(\mu_i(t)\big)^p}{\sum_{j=1}^n \big(\mu_j(t)\big)^p}, \quad \forall\ i = 1, \dots, n. \tag{4.4}$$

Compared to $\mu(\cdot)$, the portfolio $\pi^{(p)}(\cdot)$ in (4.4) decreases the proportion(s) held in the largest stock(s) and increases those placed in the smallest stock(s), while preserving the relative rankings of all stocks. The actual performance of this portfolio relative to the S&P500 index over a 22-year period is discussed in detail by Fernholz (2002), along with issues of practical implementation in the context of passive asset management.

We show in the Appendix that if the model $\mathcal{M}$ of (2.1)–(2.3) is weakly diverse on a finite time-horizon $[0, T]$, then $\pi^{(p)}(\cdot)$ outperforms the market portfolio $\mu(\cdot)$: starting with initial capital equal to $Z^\mu(0)$, the value $Z^{\pi^{(p)}}(\cdot)$ of the portfolio in (4.4) satisfies

$$P\Big[Z^{\pi^{(p)}}(T) > Z^\mu(T)\Big] = 1, \quad \text{provided that} \quad T \geq T_* := \frac{2}{p\varepsilon\delta} \cdot \log n; \tag{4.5}$$

and if $\mathcal{M}$ is uniformly weakly diverse over $[T_*, \infty)$ then $\pi^{(p)}(\cdot)$ is a superior long-term growth opportunity relative to the market, i.e., $\mathcal{L}^{\pi^{(p)},\mu} \geq \varepsilon\delta(1-p)/2$ a.s. in the notation of (2.11).

What conditions on the coëfficients $b(\cdot)$, $\sigma(\cdot)$ of $\mathcal{M}$ are sufficient for guaranteeing diversity, as in (4.1)? Certainly $\mathcal{M}$ cannot be diverse if $b_1(\cdot), \dots, b_n(\cdot)$ are bounded uniformly in (t, ω), or even if they satisfy a condition of the Novikov type

$$E\left[\exp\left\{\frac{1}{2}\int_0^T \Big|\Big|b(t)\Big|\Big|^2 dt\right\}\right] < \infty, \quad \forall\ T \in (0, \infty). \tag{4.6}$$

The reason is that, under the condition (4.6), the Girsanov theorem produces an equivalent probability measure Q under which the processes $X_1(\cdot), \dots, X_n(\cdot)$ in (2.1) become martingales. This proscribes (2.10), let alone the equation of (4.5), for *any* $T \in (0, \infty)$; see the Appendix for an argument in a somewhat more general context.

We shall see in Sect. 6 that diversity is ensured by a strongly negative rate of growth for the largest stock, resulting in a sufficiently strong repelling drift (e.g., a log-pole-type singularity) away from an appropriate boundary, and by non-negative growth-rates for all the other stocks. It turns out, however, that such a structure does not prohibit the familiar treatments of option pricing, hedging or portfolio optimization problems in the context of diverse markets; we elaborate on this point in Sect. 9.

5 The dynamics of ranked market-weights

A simple application of Itô's rule to the Eq. (2.4) gives

$$d\Big(\log \mu_i(t)\Big) = (\gamma_i(t) - \gamma^\mu(t))\,dt + \sum_{\nu=1}^m (\sigma_{i\nu}(t) - \sigma^\mu_\nu(t))\,dW_\nu(t), \quad i = 1, \dots, n \tag{5.1}$$

in the notation of (2.7), (2.8), or equivalently

$$\frac{d\mu_i(t)}{\mu_i(t)} = \left(\gamma_i(t) - \gamma^\mu(t) + \frac{1}{2}\tau^\mu_{ii}(t)\right) dt + \sum_{\nu=1}^m (\sigma_{i\nu}(t) - \sigma^\mu_\nu(t))\,dW_\nu(t) \tag{5.2}$$

for $i = 1, \dots, n$. Here, by analogy with (2.5), we have introduced

$$\tau^\pi_{ij}(t) := \sum_{\nu=1}^m (\sigma_{i\nu}(t) - \sigma^\pi_\nu(t))\,(\sigma_{j\nu}(t) - \sigma^\pi_\nu(t)) = a_{ij}(t) - a^\pi_i(t) - a^\pi_j(t) + a^{\pi\pi}(t), \tag{5.3}$$

the relative covariance (matrix-valued) process of an arbitrary portfolio $\pi(\cdot)$, and set

$$a^\pi_i(t) := \sum_{j=1}^n \pi_j(t) a_{ij}(t), \qquad a^{\pi\pi}(t) := \sum_{i=1}^n \sum_{j=1}^n \pi_i(t) a_{ij}(t) \pi_j(t).$$

In terms of the quantities of (5.3) we can express the excess rate of growth of (2.8) as

$$\gamma^\pi_*(t) = \frac{1}{2} \sum_{i=1}^n \pi_i(t) \tau^\pi_{ii}(t), \tag{5.4}$$

and for arbitrary portfolios $\pi(\cdot)$, $\rho(\cdot)$ we have the "numéraire-invariance" property

$$\gamma^\pi_*(t) = \frac{1}{2} \left(\sum_{i=1}^n \pi_i(t) \tau^\rho_{ii}(t) - \sum_{i=1}^n \sum_{j=1}^n \pi_i(t) \pi_j(t) \tau^\rho_{ij}(t) \right); \tag{5.5}$$

see Lemmata 1.3.4 and 1.3.6 in Fernholz (2002).

Now let us denote by $(p_t(1), \dots, p_t(n))$ the random permutation of $(1, \dots, n)$ for which

$$\mu_{p_t(k)}(t) = \mu_{(k)}(t), \qquad \text{and} \quad p_t(k) < p_t(k+1) \quad \text{if} \quad \mu_{(k)}(t) = \mu_{(k+1)}(t), \tag{5.6}$$

hold for $k = 1, \dots, n$. This means, roughly, that $p_t(k)$ is the name (index) of the stock with the k^{th} largest relative capitalization at time t, and that "ties are resolved by resorting to the lowest index". Using Itô's rule for convex functions of semimartingales it is shown in Fernholz (2001, 2002) that the ranked market-weights of (2.9) satisfy the dynamics

$$d\Big(\log \mu_{(k)}(t)\Big) = \big(\gamma_{p_t(k)}(t) - \gamma^\mu(t)\big)\,dt + \sum_{\nu=1}^{m} \big(\sigma_{p_t(k)\nu}(t) - \sigma^\mu_\nu(t)\big)\,dW_\nu(t)$$
$$+ \frac{1}{2}\cdot\Big[d\Lambda^{(k,k+1)}(t) - d\Lambda^{(k-1,k)}(t)\Big]. \qquad (5.7)$$

Here, for each $k = 1, \ldots, n-1$, the quantity $\Lambda^{(k,k+1)}(t) := \Lambda_{\log \mu_{(k)} - \log \mu_{(k+1)}}(t)$ is the local time that the non-negative semimartingale $\log\big(\mu_{(k)}/\mu_{(k+1)}\big)(\cdot)$ has accumulated at the origin by calendar time t; and we set $\Lambda_{\log \mu_{(0)} - \log \mu_{(1)}}(\cdot) \equiv 0$ and $\Lambda_{\log \mu_{(n)} - \log \mu_{(n+1)}}(\cdot) \equiv 0$.

On the event $\{\mu_{(1)}(t) > 1/2\}$ we have $\mu_{(2)}(t) < 1/2$, thus $\int_0^\infty 1_{\{\mu_{(1)}(t)>1/2\}} d\Lambda^{(1,2)}(t) = 0$. Therefore, with $k = 1$ the Eq. (5.7) reads

$$d\Big(\log \mu_{(1)}(t)\Big) = \big(\gamma_{(1)}(t) - \gamma^\mu(t)\big)\,dt + \frac{1}{2}\cdot 1_{\{\mu_{(1)}(t)\le 1/2\}}\cdot d\Lambda^{(1,2)}(t)$$
$$+ \sqrt{\tau^\mu_{(11)}(t)}\cdot dB(t) \qquad (5.8)$$

where $B(\cdot)$ is standard Brownian motion and

$$\gamma_{(k)}(t) := \gamma_{p_t(k)}(t), \quad \tau^\mu_{(kk)}(t) := \tau^\mu_{ii}(t)\Big|_{i=p_t(k)}. \qquad (5.9)$$

Remark 5.1 For a portfolio $\pi(\cdot)$ the conditions of (2.3) lead to the inequalities

$$\varepsilon\Big(1 - \pi_i(t)\Big)^2 \le \tau^\pi_{ii}(t) \le M(1 - \pi_i(t))(2 - \pi_i(t)) \qquad (5.10)$$

for the quantities of (5.3), and in the case of the market-portfolio to

$$\varepsilon\Big(1 - \mu_{(1)}(t)\Big)^2 \le \tau^\mu_{(kk)}(t) \le 2M, \quad t \ge 0, \;\; k = 1, \ldots, n. \qquad (5.11)$$

On the other hand, we show in the Appendix that the inequalities of (2.3) imply the bounds

$$\frac{\varepsilon}{2}\Big(1 - \pi_{(1)}(t)\Big) \le \gamma^\pi_*(t) \le M\Big(1 - \pi_{(1)}(t)\Big), \quad 0 \le t < \infty \qquad (5.12)$$

in the notation of (2.8), (2.9).

6 Ensuring diversity

Suppose that we select a number $\delta \in \Big(0, 1 - \mu_{(1)}(0)\Big)$ where $\mu_{(1)}(0) = \max_{1\le i\le n} X_i(0)/(X_1(0) + \cdots + X_n(0))$, and ask under what conditions we might have

$$\mu_{(1)}(t) < 1 - \delta, \quad \forall\; 0 \le t < \infty \qquad (6.1)$$

almost surely; this condition implies the requirement (4.1) of diversity on any finite time-horizon $[0, T]$. To simplify the analysis we shall assume $\frac{1}{2} \leq \mu_{(1)}(0) < 1 - \delta$ and consider

$$R := \inf \left\{ t \geq 0 \middle| \mu_{(1)}(t) \leq \frac{1}{2} \right\}, \qquad S := \inf \left\{ t \geq 0 | \mu_{(1)}(t) \geq 1 - \delta \right\}, \quad (6.2)$$

as well as the stopping times

$$S_k := \inf \left\{ t \geq 0 | \mu_{(1)}(t) \geq 1 - \delta_k \right\}, \qquad \delta_k = \delta + \frac{1}{k} \quad (6.3)$$

for all $k \in \mathbf{N}$ sufficiently large. For diversity, it will be enough to guarantee

$$\overline{\lim}_{k\to\infty} P\left[S_k < R\right] = 0; \quad (6.4)$$

because then $P[S < R] \leq \overline{\lim}_{k\to\infty} P\left[S_k < R\right] = 0$, and this leads to (6.1).

Theorem 6.1 *Suppose that on the event* $\left\{ \frac{1}{2} \leq \mu_{(1)}(t) < 1 - \delta \right\}$ *we have*

$$\gamma_{(k)}(t) \geq 0 \geq \gamma_{(1)}(t), \quad \forall\ k = 2, \ldots, n \quad (6.5)$$

$$\min_{2 \leq k \leq n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \geq \frac{M}{\delta Q(t)}, \ \ \textit{where}\ \ Q(t) := \log\left(\frac{1-\delta}{\mu_{(1)}(t)} \right). \quad (6.6)$$

Then (6.4), (6.1) are satisfied. On any given, finite time-horizon $[0, T]$ *the market is diverse and* $\int_0^T Q^{-2}(t)dt < \infty$ *holds a.s.*

Remark 6.1 The condition (6.6) holds, in particular, if all stocks but the largest have non-negative growth rates, whereas the growth rate of the largest stock is negative and exhibits a log-pole-type singularity as the relative capitalization of the largest stock approaches $1 - \delta$: namely, $\gamma_{(1)}(t) \leq -\frac{M}{\delta Q(t)}$ on the event $\{1/2 \leq \mu_{(1)}(t) < 1 - \delta\}$.

Remark 6.2 In terms of our market-model $\mathcal{M}$ of Sect. 2 we may specify, for instance, a constant volatility matrix $\sigma = \{\sigma_{i\nu}\}_{1 \leq i \leq n, 1 \leq \nu \leq m}$ with the properties (2.3) and a vector $g = (g_1, \ldots, g_n)'$ of non-negative numbers, and impose (2.4) in the form of a system

$$d\Big(\log X_i(t)\Big) = \left\{ g_i \cdot 1_{\mathcal{O}_i^c}(X(t)) - \frac{M}{\delta} \cdot \frac{1_{\mathcal{O}_i}(X(t))}{\log\left(\frac{1-\delta}{X_i(t)} \sum_{j=1}^n X_j(t) \right)} \right\} dt + \sum_{\nu=1}^m \sigma_{i\nu} dW_\nu(t) \quad (6.7)$$

of stochastic differential equations for the vector of stock-capitalization processes $X(\cdot) = \big(X_1(\cdot), \ldots, X_n(\cdot)\big)'$. We are using here the notation

$$\mathcal{O}_1 := \left\{ x \in (0,\infty)^n | x_1 \geq \max_{2 \leq j \leq n} x_j \right\}, \quad \mathcal{O}_n := \left\{ x \in (0,\infty)^n | x_n > \max_{1 \leq j \leq n-1} x_j \right\},$$

$$\mathcal{O}_i := \left\{ x \in (0,\infty)^n | x_i > \max_{1 \le j \le i-1} x_j, x_i \ge \max_{i+1 \le j \le n} x_j \right\}, \quad \text{for } i = 2, \dots, n-1$$

in order to keep track of the name of the stock with the largest capitalization in accordance with the convention of (5.6): $X(t) \in \mathcal{O}_i \Leftrightarrow p_t(1) = i$. With this specification all stocks but the largest behave like geometric Brownian motions (with growth rates $g_i \ge 0$ as long as $i \ne p_t(1)$, and variances $\sum_{\nu=1}^{m} \sigma_{i\nu}^2$), whereas the log-capitalization of the largest stock is subjected to a log-pole-type singularity in its drift, away from an appropriate right-boundary. Standard theory (see Veretennikov 1981) guarantees that the system of (6.7) has a pathwise unique, strong solution $X(\cdot)$ on each interval $[0, S_k]$, for all $k \in \mathbf{N}$ sufficiently large, and thus also on $[0, S) = [0, \infty)$ by the Theorem. The Eq. (6.7) prescribe rates

$$b_i(t) = \frac{1}{2} a_{ii} + g_i \cdot 1_{\mathcal{O}_i^c}(X(t)) - \frac{M}{\delta} \cdot \frac{1_{\mathcal{O}_i}(X(t))}{\log \left(\frac{1-\delta}{X_i(t)} \sum_{j=1}^{n} X_j(t) \right)}, \quad i = 1, \cdots, n$$

for the model of (2.1), (2.5). From the last assertion of Theorem 6.1 these rates satisfy $\sum_{i=1}^{n} \int_0^T (b_i(t))^2 dt < \infty$ a.s., which is the requirement (2.2).

Proof of Theorem 6.1 On the event $\left\{ \frac{1}{2} \le \mu_{(1)}(t) < 1 - \delta \right\}$ under consideration the conditions of (6.5) and (6.6) lead to

$$\begin{aligned} \gamma^{\mu}(t) - \gamma_{(1)}(t) &= \sum_{k=1}^{n} \mu_{(k)}(t) \gamma_{(k)}(t) - \gamma_{(1)}(t) + \gamma_*^{\mu}(t) \qquad (6.8) \\ &= \sum_{k=2}^{n} \mu_{(k)}(t) \gamma_{(k)}(t) - \left(1 - \mu_{(1)}(t)\right) \gamma_{(1)}(t) + \gamma_*^{\mu}(t) \\ &\ge \left(1 - \mu_{(1)}(t)\right) \left(\min_{2 \le k \le n} \gamma_{(k)}(t) - \gamma_{(1)}(t) \right) + \frac{\varepsilon}{2} \cdot \left(1 - \mu_{(1)}(t)\right) \\ &\ge \delta \left[\min_{2 \le k \le n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \right] \ge \frac{M}{Q(t)}, \end{aligned}$$

almost surely, with the help of (5.4), (5.12) and (6.1). For the process $Q(\cdot)$ of (6.6) we have from Itô's rule and (5.8) the semimartingale decomposition

$$\begin{aligned} d(\log Q(t)) = &\frac{1}{Q(t)} \left(\gamma^{\mu}(t) - \gamma_{(1)}(t) - \frac{\tau_{(11)}^{\mu}(t)}{2Q(t)} \right) dt \\ &- \frac{\sqrt{\tau_{(11)}^{\mu}(t)}}{Q(t)} dB(t) + \frac{1_{\{\mu_{(1)}(t) \le 1/2\}}}{Q(t)} d\Lambda^{(1,2)}(t); \qquad (6.9) \end{aligned}$$

in conjunction with (6.8) and the second inequality in (5.11), this gives for all integers ℓ and k large enough:

$$\begin{aligned}\log\frac{Q(\ell\wedge R\wedge S_k)}{Q(0)} &\geq \int_0^{\ell\wedge R\wedge S_k}\left(\frac{2M-\tau^{\mu}_{(11)}(t)}{2Q^2(t)}\right)dt \\ &\quad -\int_0^{\ell\wedge T\wedge S_k}\frac{1}{Q(t)}\sqrt{\tau^{\mu}_{(11)}(t)}\cdot dB(t) \\ &\geq -\int_0^{\ell\wedge R\wedge S_k}\frac{1}{Q(t)}\sqrt{\tau^{\mu}_{(11)}(t)}\cdot dB(t), \quad \text{a.s.}\end{aligned} \tag{6.10}$$

Now let us take expectations in (6.10). On the event $\{t\leq R\wedge S_k\}$ we have

$$\varepsilon\delta^2\leq\tau^{\mu}_{(11)}(t)\leq 2M,\quad \log\left(\frac{1-\delta}{1-\delta_k}\right)\leq Q(t)\leq\log\left(\frac{1-\delta}{1/2}\right)$$

from (5.11) and (6.1)–(6.3), (6.6). These bounds imply that the expectation of the stochastic integral is equal to zero. We are led to the inequalities

$$\begin{aligned}\log(Q(0)) &\leq E\left[\log\left(Q\left(\ell\wedge R\wedge S_k\right)\right)\right] \\ &\leq \log\log\left(\frac{1-\delta}{1-\delta_k}\right)\cdot P\left[S_k<\ell\wedge R\right]+\log\log\left(\frac{1-\delta}{1/2}\right)\cdot P\left[\ell\wedge R\leq S_k\right],\end{aligned}$$

and letting $\ell\to\infty$ we obtain

$$\begin{aligned}-\log\log\left(\frac{1-\delta}{1-\delta_k}\right)\cdot P\left[S_k<R\right] &\leq -\log\log\left(\frac{1-\delta}{\mu_{(1)}(0)}\right) \\ &\quad +\log\log\left(2(1-\delta)\right)\cdot P\left[R\leq S_k\right].\end{aligned} \tag{6.11}$$

This inequality is valid for all $k\in\mathbf{N}$ sufficiently large. Finally, we divide by the number $-\log\log\left(\frac{1-\delta}{1-\delta_k}\right)>0$ in (6.11), and then let $k\to\infty$; the desired conclusion (6.4) follows.

Now from (6.9) the quadratic variation of the semimartingale $\log Q(\cdot)$ satisfies

$$\varepsilon\delta^2\int_0^T\frac{1}{Q^2(t)}dt<\int_0^T\frac{\tau^{\mu}_{(11)}(t)}{Q^2(t)}dt=\langle\log Q\rangle(T)<\infty,\quad \text{a.s.}$$

in conjunction with (5.11) and (6.1), and the last claim of the theorem follows. □

The part of this proof leading up to (6.11) is similar to the argument used to establish the non-attainability of the origin by Brownian motion in dimension $n\geq 2$; see, for instance, pp. 161–162 in Karatzas and Shreve (1991). The fact that a pole-type singularity creates opportunities for relative arbitrage is reminiscent of a well-known example due to A.V. Skorohod (e.g., Karatzas and Shreve 1998, p. 11), or of the work by Delbaen and Schachermayer (1995) and by Levental and Skorohod (1995).

Remark 6.3 The inequality of condition (6.6) can be replaced by

$$\min_{2\le k\le n} \gamma_{(k)}(t) - \gamma_{(1)}(t) + \frac{\varepsilon}{2} \;\ge\; \frac{M}{\delta}\cdot F(Q(t)), \tag{6.12}$$

where $F : (0,\infty) \to (0,\infty)$ is a continuous function with the property that the associated scale function

$$U(x) := \int_1^x \exp\left[-\int_1^y F(z)dz\right] dy, \qquad 0 < x < \infty \tag{6.13}$$

satisfies $U(0+) = -\infty$. For instance, $U(x) = \log x$ when $F(x) = 1/x$ as in (6.6) or (6.8).

The function $U(\cdot)$ of (6.13) is of class $\mathcal{C}^2\big(0,\infty\big)$, so we can apply Itô's rule to the process $U(Q(t))$, $0 \le t < S$ as in (6.9). Using the strict increase and strict concavity properties $U'(\cdot) > 0, U''(\cdot) < 0$ of the scale function in (6.13), as well as the equation $U''(\cdot) + F(\cdot)U'(\cdot) = 0$, we can now repeat the steps of the argument that leads to the analogue

$$-U\left(\log\frac{1-\delta}{1-\delta_k}\right)\cdot P\left[S_k<R\right] \le -U\left(\log\frac{1-\delta}{\mu_{(1)}(0)}\right) + U\Big(\log(2(1-\delta))\Big)\cdot P\left[R\le S_k\right]$$

of (6.11), and hence to (6.4) with the help of the requirement $U(0+) = -\infty$.

7 An asymptotically weakly diverse market

Suppose we have a two-stock market model of the form

$$dX_i(t) = X_i(t)\left[b_i(t)dt + \frac{1}{\sqrt{2}}dW_i(t)\right], \quad X_i(0) = x \in (0,\infty) \quad \text{for } i = 1,2 \tag{7.1}$$

driven by the planar Brownian motion $W = (W_1, W_2)$. Then $W := \frac{1}{\sqrt{2}}\big(W_2 - W_1\big)$ is standard Brownian motion, and we have

$$X_2(t) = X_1(t)\cdot \exp(Z(t)), \text{ where } Z(t) := \int_0^t (b_2(s) - b_1(s))ds + W(t), \tag{7.2}$$

$$\mu_1(t) = \frac{X_1(t)}{X_1(t) + X_2(t)} = \frac{1}{1+e^{Z(t)}}, \quad \mu_2(t) = \frac{1}{1+e^{-Z(t)}},$$
$$\text{thus } \mu_{(1)}(t) = \frac{1}{1+e^{-|Z(t)|}} \tag{7.3}$$

for $0 \le t < \infty$. Now let us select $b_1(\cdot) \equiv 0$ and $b_2(\cdot) \equiv -\alpha Z(\cdot)1_{[1,\infty)}(\cdot)$ for a suitable real constant $\alpha > 0$ to be determined below. With these choices the process $Z(\cdot)$ of (7.2) becomes $Z(t) = W(t)$ for $0 \le t \le 1$ and

$$Z(t) = W(1) - \alpha\int_1^t Z(s)ds + \widetilde{W}(t) \quad \text{for } 1 \le t < \infty, \tag{7.4}$$

where $\{\widetilde{W}(t) := W(t) - W(1),\ 1 \le t < \infty\}$ is standard Brownian motion and independent of $Z(1) = W(1)$. In other words, the process $\{Z(t),\ 1 \le t < \infty\}$ is Ornstein-Uhlenbeck, with gaussian initial distribution $\mathcal{N}(0,1)$ and gaussian invariant distribution $\mathcal{N}(0, 1/2\alpha)$; see Karatzas and Shreve (1991), p. 358 for the latter assertion. With the choice $\alpha = 1/2$ the process $Z(\cdot)$ is stationary, and its ergodic behavior gives

$$\lim_{T\to\infty} \frac{1}{T}\int_1^{T+1} \mu_{(1)}(t)dt = \lim_{T\to\infty} \frac{1}{T}\int_0^T \frac{dt}{1+e^{-|Z(t+1)|}}$$
$$= E\left(\frac{1}{1+e^{-|Z(1)|}}\right) < 1-\delta, \qquad \text{a.s.}$$

for any $0 < \delta < E\left(\frac{e^{-|Z(1)|}}{1+e^{-|Z(1)|}}\right) = \sqrt{\frac{2}{\pi}}\int_0^\infty \frac{e^{-z}}{1+e^{-z}} e^{-z^2/2}dz$. Thus, the model $\mathcal{M}$ of (7.1) is asymptotically weakly diverse.

However, *diversity fails for this model*. For any $T \in [1,\infty)$ and $\delta \in (0,\infty)$ we have

$$P\left[\mu_{(1)}(T) \ge 1-\delta\right] = P\left[|Z(T)| \ge K\right] = \frac{2}{\sqrt{2\pi}}\int_K^\infty e^{-u^2/2}du > 0,$$

where $K := \log(1-\delta) - \log\delta$. In fact, *weak diversity fails as well*. For an arbitrary $T \in (1,\infty)$ and $\delta \in (0,1)$, select $\varepsilon \in (0,T)$ and $\zeta > 0$ so that $\delta \ge \frac{(\varepsilon/T)+e^{-\zeta}}{1+e^{-\zeta}}$; then it is straightforward that the event $A_{\varepsilon,\zeta} := \{\inf_{\varepsilon\le t\le T} |Z(t)| \ge \zeta\}$ has positive probability $P(A_{\varepsilon,\zeta}) > 0$ and that

$$\frac{1}{T}\int_\varepsilon^T \mu_{(1)}(t)dt = \frac{1}{T}\int_\varepsilon^T \frac{dt}{1+e^{-|Z(t)|}} \ge \frac{T-\varepsilon}{T(1+e^{-\zeta})} \ge 1-\delta$$

holds a.e. on $A_{\varepsilon,\zeta}$, thus leading to $P\left(\int_0^T \mu_{(1)}(t)dt \ge (1-\delta)T\right) > 0$. It can be shown that the model of (7.1) admits a unique equivalent martingale measure.

Remark 7.1 The examples of Sect. 6 can be easily modified to produce a model $\mathcal{M}$ which is weakly diverse but not diverse. Indeed, let us start by considering a model $\mathcal{M}^{(2\delta)}$ with constant volatilities σ_{ij} and with rates of return $b_i^{(2\delta)}(\cdot),\ i = 1,\cdots,n$ such that $P\big(\mu_{(1)}(t) < 1-2\delta,\ \forall 0 \le t \le T\big) = 1$ is satisfied for some $T \in (0,\infty)$ and $\delta \in (0,1/4)$. The idea is to divide the time-horizon $[0,T]$ into the two intervals $[0,T/2)$ and $[T/2,T]$, select $\eta \in (2\delta, 1/2)$, and set

$$b_i(t) := b_i^{(2\delta)}(t) \cdot 1_{\{S\le t\le T,\ S\le T/2\}}, \quad \text{where } S := \inf\{t \ge 0 | \mu_{(1)}(t) \ge 1-\eta\} \wedge T. \tag{7.5}$$

We claim that the model $\mathcal{M}$, with volatilities σ_{ij} and rates of return given by (7.5), is weakly diverse on $[0,T]$. To see this, consider two cases: For $\omega \in \{S \le T/2\}$ the recipe (7.5) and (4.1) guarantee $\mu_{(1)}(t,\omega) < 1-2\delta < 1-\delta, \forall 0 \le t \le T$; and for $\omega \in \{S > T/2\}$ we have

$$\frac{1}{T}\int_0^T \mu_{(1)}(t,\omega)dt \le \frac{1}{T}\int_0^{T/2}(1-\eta)dt + \frac{1}{T}\int_{T/2}^T 1\cdot dt = 1-(\eta/2) < 1-\delta.$$

But for this $\mathcal{M}$ the property (4.1) fails: the event $B := \{S > T/2\}$ has positive probability, and with $A := \{\max_{0\le t\le T} \mu_{(1)}(t) \ge 1-\delta\}$ we have $P(A\cap B) > 0$. To see this, consider the special case $n = 2$, $\sigma_{12} = \sigma_{21} = 0$, $\sigma_{11} = \sigma_{22} = 1/\sqrt{2}$ as in (7.1), and observe that on the event $B = \{S > T/2\}$ we have $Z(\cdot) \equiv W(\cdot)$ in (7.2) and

$$\max_{0\le t\le T} \mu_{(1)}(t) \ge 1-\delta \qquad \Longleftrightarrow \qquad \max_{0\le t\le T} |Z(t)| \ge K := \log\left(\frac{1-\delta}{\delta}\right).$$

Consequently,

$$\begin{aligned} P(A\cap B) &= P\Big[\max_{0\le t\le T} |W(t)| \ge K; S > T/2\Big] \\ &\ge P\Big[\max_{T/2\le t\le T} |W(t) - W(T/2)| \ge 2K; S > T/2\Big] \\ &= P\left(\max_{T/2\le t\le T} |W(t) - W(T/2)| \ge 2K \,\Big|\, S > T/2\right) \cdot P(S > T/2) \\ &\ge P\left(\max_{0\le t\le T/2} |W(t)| \ge 2K\right) \cdot P(S > T/2) > 0, \end{aligned}$$

since $\{W(t) - W(T/2); T/2 \le t < \infty\}$ is a Brownian motion and independent of $\mathcal{F}(T/2)$, a $\sigma-$algebra that contains the event $\{S > T/2\}$.

8 Mirror portfolios, short-horizon relative arbitrage

We saw in (4.5) that, in weakly diverse markets and over sufficiently long time-horizons, there exist portfolios (e.g., the diversity-weighted portfolio $\pi^{(p)}(\cdot)$ of (4.4)) that represent arbitrage opportunities relative to the market portfolio $\mu(\cdot)$. We shall show in this section that relative arbitrage can be constructed on arbitrary time-horizons; there always exist portfolios that constistently outperform or underperform a weakly diverse market.

In order to do this we have to introduce the notion of **extended portfolio**: a progressively measurable and uniformly bounded process $\pi(\cdot) = (\pi_1(\cdot), \cdots, \pi_n(\cdot))'$ with values in $\Delta^n = \{(\pi_1, \cdots, \pi_n) \in I\!R^n | \sum_{i=1}^n \pi_i = 1\}$. In other words, an extended portfolio can sell one or more stocks short, but certainly not all. By contrast, the portfolios of Sect. 2 are "all-long" portfolios: they allow no short-selling.

Let us fix a baseline portfolio $m(\cdot)$; this will typically, though not necessarily, be the market portfolio $\mu(\cdot)$. For any extended portfolio $\pi(\cdot)$ and any fixed real number $p \ne 0$ we define the $p-$mirror-image of $\pi(\cdot)$ with respect to $\mu(\cdot)$ by

$$\widetilde{\pi}^{(p)}(\cdot) := p\pi(\cdot) + (1-p)m(\cdot). \tag{8.1}$$

This is clearly an extended portfolio, and a portfolio in the strict ("all-long") sense of Sect. 2 if this is the case for $\pi(\cdot)$ and $0 < p < 1$. If $p = -1$ we call $\widetilde{\pi}^{(-1)}(\cdot) = 2m(\cdot) - \pi(\cdot)$ the "mirror image" of $\pi(\cdot)$ with respect to $m(\cdot)$. We notice

$$\left(\widetilde{\pi}^{(p)}\right)^{\sim(q)} = \widetilde{\pi}^{(pq)}, \qquad \left(\widetilde{\pi}^{(p)}\right)^{\sim(1/p)} = \pi. \tag{8.2}$$

Let us recall the notation $\tau^m(\cdot) = \{\tau_{ij}^m(\cdot)\}_{1\le i,j\le n}$ of (5.3) for the matrix-valued covariance process of $m(\cdot)$, define the *relative covariance of* $\pi(\cdot)$ *with respect to* $m(\cdot)$ by

$$\tau_{\pi\pi}^m(t) := (\pi(t) - m(t))' a(t) (\pi(t) - m(t)) \ge \varepsilon ||\pi(t) - m(t)||^2, \tag{8.3}$$

and make the elementary observations

$$\tau^m(\cdot)m(\cdot) \equiv 0, \quad \tau_{\pi\pi}^m(\cdot) = \pi'(\cdot)\tau^m(\cdot)\pi(\cdot) = \tau_{mm}^\pi(\cdot), \quad \tau_{\widetilde{\pi}^{(p)}\widetilde{\pi}^{(p)}}^m(\cdot) = p^2 \cdot \tau_{\pi\pi}^m(\cdot). \tag{8.4}$$

We shall take $m(\cdot) \equiv \mu(\cdot)$ from now on. The relative performance of $\pi(\cdot)$ with respect to $\mu(\cdot)$ is given in (1.2.16) of Fernholz (2002) by

$$d\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) = \sum_{i=1}^n (\pi_i(t) - \mu_i(t)) d\log\mu_i(t) + (\gamma_*^\pi(t) - \gamma_*^\mu(t))\, dt. \tag{8.5}$$

Writing this expression for $\widetilde{\pi}^{(p)}(\cdot)$ in place of $\pi(\cdot)$, recalling $\widetilde{\pi}^{(p)} - \mu = p(\pi - \mu)$ from (8.1), and then subtracting (8.5) multiplied by p, we obtain

$$d\log\left(\frac{Z^{\widetilde{\pi}^{(p)}}(t)}{Z^\mu(t)}\right) = p \cdot d\log\left(\frac{Z^\pi(t)}{Z^\mu(t)}\right) + (p-1)\gamma_*^\mu(t)dt + \left(\gamma_*^{\widetilde{\pi}^{(p)}}(t) - p\gamma_*^\pi(t)\right) dt. \tag{8.6}$$

But now recall the expressions of (5.5), (8.4) and (5.4), to obtain

$$\begin{aligned} 2\left(\gamma_*^{\widetilde{\pi}^{(p)}}(t) - p\gamma_*^\pi(t)\right) &= \sum_{i=1}^n \left(\widetilde{\pi}_i^{(p)}(t) - p\pi_i(t)\right) \tau_{ii}^\mu(t) - \tau_{\widetilde{\pi}^{(p)}\widetilde{\pi}^{(p)}}^\mu(t) + p\tau_{\pi\pi}^\mu(t) \\ &= (1-p) \cdot \sum_{i=1}^n \mu_i(t)\tau_{ii}^\mu(t) + p\tau_{\pi\pi}^\mu(t) - p^2\tau_{\pi\pi}^\mu(t) \\ &= (1-p) \cdot \left[2\gamma_*^\mu(t) + p\tau_{\pi\pi}^\mu(t)\right]. \end{aligned}$$

Substituting back into (8.6) we get

$$\log\left(\frac{Z^{\widetilde{\pi}^{(p)}}(T)}{Z^\mu(T)}\right) = p \cdot \log\left(\frac{Z^\pi(T)}{Z^\mu(T)}\right) + \frac{p(1-p)}{2}\int_0^T \tau_{\pi\pi}^\mu(t)dt \tag{8.7}$$

and note that the last term is non-negative, by (8.3).

Lemma 8.1 *Suppose that the extended portfolio* $\pi(\cdot)$ *is such that the conditions*

$$P\left(\frac{Z^\pi(T)}{Z^\mu(T)} \ge \beta\right) = 1 \quad \text{or} \quad P\left(\frac{Z^\pi(T)}{Z^\mu(T)} \le \frac{1}{\beta}\right) = 1 \tag{8.8}$$

and

$$P\left(\int_0^T \tau_{\pi\pi}^\mu(t)dt \ge \eta\right) = 1 \tag{8.9}$$

hold, for some $\beta > 0$ and $\eta > 0$. Then there exists an extended portfolio $\widehat{\pi}(\cdot)$ such that

$$P\left(Z^{\widehat{\pi}}(T) < Z^{\mu}(T)\right) = 1. \tag{8.10}$$

Remark 8.1 Condition (8.8) postulates that the extended portfolio $\pi(\cdot)$ is "not very different" from the market portfolio. But condition (8.9) mandates that $\pi(\cdot)$ "must be sufficiently different" from the market portfolio; indeed, $\int_0^T \tau_{\pi\pi}^{\mu}(t)dt \geq \varepsilon \sum_{i=1}^n \int_0^T |\pi_i(t) - \mu_i(t)|^2 dt$ from (8.3), so (8.9) holds if the expression $||\pi - \mu||_{\mathbf{L}^2([0,T])}$ is bounded away from zero, a.s.

Proof of Lemma 8.1 If we have $P\left[(Z^{\pi}(T)/Z^{\mu}(T)) \leq 1/\beta\right] = 1$, then it suffices to take $p > 1 + (2/\eta) \cdot \log(1/\beta)$ and observe from (8.9), (8.7) that $\widehat{\pi}(\cdot) \equiv \widetilde{\pi}^{(p)}(\cdot)$ satisfies

$$\log\left(\frac{Z^{\widehat{\pi}}(T)}{Z^{\mu}(T)}\right) \leq p \cdot \left[\log\left(\frac{1}{\beta}\right) + \frac{\eta}{2}(1-p)\right] < 0, \quad \text{a.s.}$$

If on the other hand we have $P\left[(Z^{\pi}(T)/Z^{\mu}(T)) \geq \beta\right] = 1$, then it suffices to take $p < \min\bigl(0, 1 - (2/\eta) \cdot \log(1/\beta)\bigr)$ and observe from (8.7) that $\widehat{\pi}(\cdot) \equiv \widetilde{\pi}^{(p)}(\cdot)$ satisfies

$$\log\left(\frac{Z^{\widehat{\pi}}(T)}{Z^{\mu}(T)}\right) \leq p \cdot \left[-\log\left(\frac{1}{\beta}\right) + \frac{\eta}{2}(1-p)\right] < 0, \quad \text{a.s.}$$

□

Example 8.1 With $\pi = e_1 = (1, 0, \cdots, 0)'$ and $m(\cdot) \equiv \mu(\cdot)$ the market portfolio, take a number $p > 1$ (to be detemined in a moment) and define the extended portfolio

$$\widehat{\pi}(t) := \widetilde{\pi}^{(p)}(t) = pe_1 + (1-p)\mu(t), \quad 0 \leq t < \infty, \tag{8.11}$$

which takes a long position in the first stock and a short position in the market. (This is not a very easy portfolio to implement in actual practice.) In particular, $\widehat{\pi}_1(t) = p + (1-p)\mu_1(t)$ and $\widehat{\pi}_i(t) = (1-p)\mu_i(t)$ for $i = 2, \cdots, n$. Then we have

$$\log\left(\frac{Z^{\widehat{\pi}}(T)}{Z^{\mu}(T)}\right) = p \cdot \left[\log\left(\frac{\mu_1(T)}{\mu_1(0)}\right) - \frac{p-1}{2}\int_0^T \tau_{11}^{\mu}(t)dt\right] \tag{8.12}$$

from (8.7). But taking $\beta := \mu_1(0)$ we have $(\mu_1(T)/\mu_1(0)) \leq 1/\beta$, and if the market is weakly diverse on $[0, T]$ we obtain from (5.10) and the Cauchy-Schwarz inequality

$$\int_0^T \tau_{11}^{\mu}(t)dt \geq \varepsilon \int_0^T \left(1 - \mu_{(1)}(t)\right)^2 dt > \varepsilon\delta^2 T =: \eta. \tag{8.13}$$

From Lemma 8.1 the market portfolio represents then an arbitrage opportunity with respect to the extended portfolio $\widehat{\pi}(\cdot)$ of (8.11), provided that for any given $T \in$

$(0,\infty)$ we select $p > p(T) := 1 + \frac{2}{\varepsilon\delta^2 T} \cdot \log\left(\frac{1}{\mu_1(0)}\right)$. Note that $\lim_{T\downarrow 0} p(T) = \infty$. □

The extended portfolio $\widehat{\pi}(\cdot)$ of (8.11) can be used to create all-long portfolios that underperform (Example 8.2) or outperform (Example 8.3) the market portfolio $\mu(\cdot)$, over any given time-horizon $T \in (0,\infty)$. The idea is to embed $\widehat{\pi}(\cdot)$ in a sea of market portfolio, swamping the short positions while retaining the essential portfolio characteristics. Crucial in these constructions is the a.s. comparison

$$Z^{\widehat{\pi}}(t) \le \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^p \cdot Z^{\mu}(t), \quad 0 \le t < \infty, \tag{8.14}$$

a direct consequence of (8.12). Here and in what follows we assume $Z^{\mu}(0) = Z^{\widehat{\pi}}(0) = 1$.

Example 8.2 Consider an investment strategy $\rho(\cdot)$ that places one dollar in the portfolio $\widehat{\pi}(\cdot)$ of (8.11) and $(p-1)/(\mu_1(0))^p$ dollars in the market portfolio $\mu(\cdot)$ at time $t = 0$, and makes no change afterwards. The number p is chosen as in Example 8.1. The value $Z^{\rho}(\cdot)$ of this strategy is clearly $Z^{\rho}(t) = Z^{\widehat{\pi}}(t) + \frac{p-1}{(\mu_1(0))^p} \cdot Z^{\mu}(t) > 0$, $0 \le t < \infty$, and is generated by the extended portfolio with weights

$$\rho_i(t) = \frac{1}{Z^{\rho}(t)}\left[\widehat{\pi}_i(t) \cdot Z^{\widehat{\pi}}(t) + \frac{p-1}{(\mu_1(0))^p} \cdot \mu_i(t) Z^{\mu}(t)\right], \quad \text{for } i = 1, \cdots, n.$$

Clearly $\sum_{i=1}^n \rho_i(t) = 1$; and since both $\widehat{\pi}_1(t)$ and $\mu_1(t)$ are positive, we have $\rho_1(t) > 0$ as well. To check that $\rho(\cdot)$ is an all-long portfolio, observe that the dollar amount it invests at time t in any stock $i = 2, \cdots, n$ is

$$-(p-1)\mu_i(t) \cdot Z^{\widehat{\pi}}(t) + \frac{p-1}{(\mu_1(0))^p} \cdot \mu_i(t) Z^{\mu}(t) \geqslant \frac{(p-1)\mu_i(t)}{(\mu_1(0))^p}\left[1 - (\mu_1(t))^p\right] Z^{\mu}(t) > 0$$

thanks to (8.14). On the other hand, $\rho(\cdot)$ *underperforms* at $t = T$ a market portfolio that starts out with the same initial capital $z := Z^{\rho}(0) = 1 + (p-1)/(\mu_1(0))^p$, since $\rho(\cdot)$ holds a mix of $\mu(\cdot)$ and $\widehat{\pi}(\cdot)$, and $\widehat{\pi}(\cdot)$ underperforms the market at $t = T$:

$$Z^{\rho}(T) = Z^{\widehat{\pi}}(T) + \frac{p-1}{(\mu_1(0))^p} Z^{\mu}(T) < z Z^{\mu}(T) = Z^{z,\mu}(T) \quad \text{a.s., from (8.10).}$$

Example 8.3 Now consider a strategy $\eta(\cdot)$ that invests $p/(\mu_1(0))^p$ dollars in the market portfolio and -1 dollar in $\widehat{\pi}(\cdot)$ at time $t = 0$, and makes no change thereafter. The number $p > 1$ is chosen again as in Example 8.1. The value $Z^{\eta}(\cdot)$ of this strategy is

$$Z^{\eta}(t) = \frac{p}{(\mu_1(0))^p} \cdot Z^{\mu}(t) - Z^{\widehat{\pi}}(t) \ge \frac{Z^{\mu}(t)}{(\mu_1(0))^p}\left[p - (\mu_1(t))^p\right] > 0, \quad 0 \le t < \infty \tag{8.15}$$

thanks to (8.14) and $p > 1 > (\mu_1(t))^p$. As before, $Z^{\eta}(\cdot)$ is generated by an extended portfolio $\eta(\cdot)$ with weights

$$\eta_i(t) = \frac{1}{Z^{\eta}(t)}\left[\frac{p\mu_i(t)}{(\mu_1(0))^p} \cdot Z^{\mu}(t) - \widehat{\pi}_i(t) \cdot Z^{\widehat{\pi}}(t)\right], \quad i = 1, \cdots, n \tag{8.16}$$

that clearly satisfy $\sum_{i=1}^n \eta_i(t) = 1$. Now for $i = 2, \cdots, n$ we have $\widehat{\pi}_i(t) = -(p-1)\mu_i(t) < 0$, so $\eta_2(\cdot), \ldots, \eta_n(\cdot)$ are strictly positive. To check that $\eta(\cdot)$ is actually an all-long portfolio, it remains to verify $\eta_1(t) \geq 0$; but the dollar amount

$$\frac{p\mu_1(t)}{(\mu_1(0))^p} \cdot Z^\mu(t) - \big[p - (p-1)\mu_1(t)\big] \cdot Z^{\widehat{\pi}}(t)$$

invested by $\eta(\cdot)$ in the first stock at time t, dominates $\frac{p\mu_1(t)}{(\mu_1(0))^p} \cdot Z^\mu(t) - \big[p - (p-1)\mu_1(t)\big] \cdot \left(\frac{\mu_1(t)}{\mu_1(0)}\right)^p Z^\mu(t)$, or equivalently the quantity

$$\frac{Z^\mu(t)\mu_1(t)}{(\mu_1(0))^p} \cdot \Big[(p-1)\left(\mu_1(t)\right)^p + p\left\{1 - (\mu_1(t))^{p-1}\right\}\Big] > 0,$$

again thanks to (8.14) and $p > 1 > (\mu_1(t))^p$. Thus $\eta(\cdot)$ is indeed an all-long portfolio.

On the other hand, $\eta(\cdot)$ *outperforms* at $t = T$ a market portfolio with the same initial capital of $\zeta := Z^\eta(0) = p/(\mu_1(0))^p - 1 > 0$ dollars, because $\eta(\cdot)$ is long in the market $\mu(\cdot)$ and short in the extended portfolio $\widehat{\pi}(\cdot)$, which underperforms the market at $t = T$:

$$Z^\eta(T) = \frac{p}{(\mu_1(0))^p} Z^\mu(T) - Z^{\widehat{\pi}}(T) > \zeta Z^\mu(T) = Z^{\zeta,\mu}(T) \quad \text{a.s., from (8.10).}$$

9 Hedging in weakly diverse markets

Suppose now that we place a small investor in a market $\mathcal{M}$ as in (2.1)–(2.5) and allow him to invest also in a money-market with interest rate $r : [0,\infty) \times \Omega \to [0,\infty)$, a progressively measurable, locally square-integrable process. A dollar invested at time $t = 0$ in the money market grows to $B(T) = \exp\{\int_0^T r(u)du\}$ at time $t = T$.

Starting with initial capital $z > 0$, the investor can choose at any time t a **trading strategy** $\varphi(t) = (\varphi_1(t), \cdots, \varphi_n(t))'$. With $Z^{z,\varphi}(t)$ denoting the value of the strategy at time t, the quantity $\varphi_i(t)$ is the dollar amount invested in the i^{th} stock and $Z^{z,\varphi}(t) - \sum_{i=1}^n \varphi_i(t)$ the amount in the money-market. These quantities are real-valued, and any one of them may be negative: selling stock short is allowed, as is borrowing from (as opposed to depositing into) the money-market. We require only that the trading strategy $\varphi(\cdot)$ be progressively measurable and satisfy $\sum_{i=1}^n \int_0^T [(\varphi_i(t))^2 + |\varphi_i(t)||b_i(t) - r(t)|]dt < \infty$ a.s., on any given time-horizon $[0,T]$. With this understanding the value-process $Z(\cdot) \equiv Z^{z,\varphi}(\cdot)$ satisfies

$$\begin{aligned} dZ(t) &= \sum_{i=1}^n \varphi_i(t) \cdot \frac{dX_i(t)}{X_i(t)} + \left(Z(t) - \sum_{i=1}^n \varphi_i(t)\right) \cdot \frac{dB(t)}{B(t)} \qquad (9.1) \\ &= r(t)Z(t)dt + \sum_{i=1}^n \varphi_i(t)\left((b_i(t) - r(t))dt + \sum_{\nu=1}^m \sigma_{i\nu}(t)dW_\nu(t)\right) \\ &= r(t)Z(t)dt + \varphi'(t)\sigma(t)d\widehat{W}(t), \end{aligned}$$

a simple linear equation. We have introduced the processes

$$\widehat{W}(t) := W(t) + \int_0^t \vartheta(s)ds, \qquad \vartheta(t) := \sigma'(t)\big(\sigma(t)\sigma'(t)\big)^{-1}[b(t) - r(t)\mathbf{1}] \tag{9.2}$$

with $\mathbf{1} = (1, \cdots, 1)' \in I\!R^n$ and $\int_0^T ||\vartheta(t)||^2 dt < \infty$ a.s. for any $T \in (0, \infty)$; recall here the conditions (2.2), (2.3) and the local square-integrability of $r(\cdot)$. In this notation we can write the Eq. (2.1) as

$$d\left(\frac{X_i(t)}{B(t)}\right) = \left(\frac{X_i(t)}{B(t)}\right) \cdot \sum_{\nu=1}^m \sigma_{i\nu}(t) d\widehat{W}_\nu(t), \qquad i = 1, \cdots, n. \tag{9.3}$$

The solution of the Eq. (9.1) is given by

$$Z^{z,\varphi}(t)/B(t) = z + \int_0^t (\varphi'(s)/B(s))\sigma(s)d\widehat{W}(s), \quad 0 \le t < \infty. \tag{9.4}$$

We shall denote by $\varPhi_T(z)$ the class of trading strategies $\varphi(\cdot)$ that satisfy $P\big[Z^{z,\varphi}(t) \ge 0, \ \forall \ 0 \le t \le T\big] = 1$ for a given $T \in (0, \infty)$, and set $\varPhi(z) := \cap_{0<T<\infty} \varPhi_T(z)$. This class contains the extended portfolios of Sect. 8: if $\pi(\cdot)$ is an extended portfolio and $Z^\pi(\cdot)$ its value-process with initial capital $Z^\pi(0) = z > 0$, then $\varphi_i(\cdot) := \pi_i(\cdot) Z^\pi(\cdot)$, $1 \le i \le n$ defines a trading strategy, and $Z^{z,\varphi}(\cdot) \equiv Z^\pi(\cdot) > 0$ satisfies the analogue of (9.4)

$$d(Z^\pi(t)/B(t)) = (Z^\pi(t)/B(t)) \cdot \pi'(t)\sigma(t)d\widehat{W}(t).$$

Remark 9.1 If $\mathcal{M}$ is weakly diverse on some finite horizon $[0, T]$, then the process

$$L(t) := \exp\left(-\int_0^t \vartheta'(s)dW(s) - \frac{1}{2}\int_0^t ||\vartheta(s)||^2 ds\right) > 0, \quad 0 \le t < \infty \tag{9.5}$$

is a local martingale and a supermartingale, but *is not a martingale*. For if it were, then the measure $Q_T(A) := E[L(T) \cdot 1_A]$ would be a probability on $\mathcal{F}(T)$. Under this probability measure the process $\widehat{W}(\cdot)$ of (9.2) would be Brownian motion, and the discounted capitalization-processes $X_i(\cdot)/B(\cdot)$ would be martingales on the interval $[0, T]$, from (9.3), (2.3). But this would proscribe (2.10) on this interval for any two extended portfolios $\pi(\cdot)$ and $\rho(\cdot)$, contradicting (4.5) and the examples of Sect. 8 (see the Appendix for a formal argument along these lines).

Thus, in a weakly diverse market the process $L(\cdot)$ of (9.5) is a strict local martingale in the sense of Elworthy et al. (1997): we have $E[L(t)] < 1$ for every $t \in (0, \infty)$.

Remark 9.2 Because $L(\cdot)$ is a local martingale there exists an increasing sequence $\{S_k\}_{k\in\mathbf{N}}$ of stopping times with $\lim_{k\to\infty} S_k = \infty$ a.s. such that $L(\cdot \wedge S_k)$ is a martingale for every $k \in \mathbf{N}$ (for instance, take $S_k = \inf\{t \geq 0 | \int_0^t ||\vartheta(s)||^2 ds \geq k\}$). Thus, if we replace T by $T \wedge S_k$ in (2.10), this property cannot hold for *any* extended portfolios $\pi(\cdot)$ and $\rho(\cdot)$: there is no possibility for relative arbitrage on the horizon $[0, T \wedge S_k]$ for any $k \in \mathbf{N}$. But in the limit as $k \to \infty$ a relative arbitrage of the type (2.10) appears, as in (4.5) or in Example 8.1, if $\mathcal{M}$ is weakly diverse on $[0, T]$.

• The failure of the exponential process $L(\cdot)$ in (9.5) to be a martingale does not preclude, however, the possibility for hedging contingent claims in a market $\mathcal{M}$ which is weakly diverse on some finite horizon $[0, T]$. To see why, consider an $\mathcal{F}(T)$-measurable random variable $Y : \Omega \to [0, \infty)$ that satisfies

$$0 < y_0 := E[YL(T)/B(T)] < \infty. \tag{9.6}$$

If we view Y as a liability (contingent claim) that the investor faces and has to cover (hedge) at time $t = T$, the question is to characterize the smallest amount of initial capital that allows the investor to hedge this liability without risk; namely, the *hedging price*

$$h^Y := \inf\{z > 0 | \text{there exists } \varphi(\cdot) \in \Phi_T(z) \text{ such that } Z^{z,\varphi}(T) \geq Y \text{ holds a.s.}\}. \tag{9.7}$$

We proceed as in the standard treatment of this question (e.g., Karatzas and Shreve (1998), Chapt. 2) but under the probability measure P, the only one now at our disposal. From (9.1)–(9.3) and the differential equation $dL(t) = -L(t)\vartheta'(t)dW(t)$ for the exponential process $L(\cdot)$ of (9.5), we obtain that each of the processes

$$\widetilde{X}_i(t) := \frac{L(t)X_i(t)}{X_i(0)B(t)} = 1 + \int_0^t \widetilde{X}_i(s) \cdot \sum_{\nu=1}^m (\sigma_{i\nu}(s) - \vartheta_\nu(s))dW_\nu(s), \quad i = 1, \cdots, n \tag{9.8}$$

$$\widetilde{Z}^\varphi(t) := \frac{L(t)Z^{z,\varphi}(t)}{zB(t)} = 1 + \int_0^t \frac{L(s)}{zB(s)} \Big(\varphi'(s)\sigma(s) - Z^{z,\varphi}(s)\vartheta'(s)\Big) dW(s) \tag{9.9}$$

(products of $L(\cdot)$ with the discounted stock-capitalizations and with the discounted values of investment strategies in $\Phi(z)$, respectively) is a non-negative local martingale, hence a supermartingale. It is not hard to see (in the Appendix) that

$$\text{the processes } \widetilde{X}_i(\cdot),\ i = 1, \cdots, n \text{ of (9.8) are strict local martingales.} \tag{9.10}$$

In particular, $E[L(T)X_i(T)/B(T)] < X_i(0)$ holds for all $T \in (0, \infty)$. And for any $z > 0$ in the set of (9.7), there exists some $\varphi(\cdot) \in \Phi_T(z)$ such that

$$E[YL(T)/B(T)] \leq E[Z^{z,\varphi}(T)L(T)/B(T)] \leq z, \tag{9.11}$$

so $y_0 = E[YL(T)/B(T)] \leq h^Y$.

• Let us suppose from now on that $m = n$, i.e., that we have exactly as many sources of randomness as there are stocks in the market $\mathcal{M}$; that the square-matrix

$\sigma(t,\omega) = \{\sigma_{ij}(t,\omega)\}_{1\le i,j\le n}$ is invertible for every $(t,\omega) \in [0,T] \times \Omega$; and that the filtration $\mathbf{F} = \{\mathcal{F}(t)\}_{0\le t\le T}$ is generated by the Brownian motion $W(\cdot)$ itself, namely, $\mathcal{F}(t) = \sigma(W(s);\ 0 \le s \le t)$. The martingale representation property of this Brownian filtration gives

$$M(t) := E\left[\frac{YL(T)}{B(T)}\Big|\mathcal{F}(t)\right] = y_0 + \int_0^t \psi'(s)dW(s) \ge 0, \qquad 0 \le t \le T \tag{9.12}$$

for some progressively measurable process $\psi : [0,T] \times \Omega \to I\!R^n$ with $\sum_{i=1}^n \int_0^T (\psi_i(t))^2 dt < \infty$ a.s. Setting

$$\widehat{Z}(\cdot) := M(\cdot)B(\cdot)/L(\cdot), \quad \widehat{\varphi}(\cdot) := \frac{B(\cdot)}{L(\cdot)}\Big(\sigma^{-1}(\cdot)\Big)'\big(\psi(\cdot) + M(\cdot)\vartheta(\cdot)\big)$$

and comparing (9.9) with (9.12), we observe $\widehat{Z}(0) = y_0$, $\widehat{Z}(T) = Y$ and $\widehat{Z}(\cdot) \equiv Z^{y_0,\widehat{\varphi}}(\cdot) \ge 0$, almost surely.

Therefore, the trading strategy $\widehat{\varphi}(\cdot)$ is in $\Phi_T(y_0)$ and satisfies the *exact replication property* $Z^{y_0,\widehat{\varphi}}(T) = Y$ a.s. This implies that y_0 belongs to the set on the right-hand-side of (9.7), and so $y_0 \ge h^Y$. But we have already established the reverse inequality, actually in much greater generality, so for the hedging price of (9.7) we get the Black-Scholes-type formula

$$h^Y = E[YL(T)/B(T)] \tag{9.13}$$

under the assumptions of the preceding paragraph. In particular, a market $\mathcal{M}$ that is weakly diverse – hence without an equivalent probability measure under which discounted stock-prices are (at least local) martingales – can nevertheless be **complete**.

But in such a market we shall have from (9.10), (9.13) that $h^{X_1(T)} < X_1(0)$, namely, that the hedging price for the stock value (capitalization) at time $T > 0$ is strictly less than the current value $X_1(0)$: the existence of relative arbitrage has made this ‘reduction’ possible.

Remark 9.3 In order to make a connection with the classical non-arbitrage theory (e.g. Delbaen and Schachermayer 1994, 1998; Lowenstein and Willard 2000), let us note that we have constrained ourselves to trading strategies that start from a strictly positive wealth and stay strictly positive – while the classical notions of arbitrage (including the so-called *“free lunches with vanishing risk”* or *“free snacks”*) require that wealth start from zero and remain bounded from below. A slight reformulation of these arbitrage notions for our model makes them completely equivalent to the ones in the above-mentioned papers. With this in mind it can be seen that Eq. (4.5) implies the existence of a (classical) arbitrage opportunity – not in the original market, but in one where the market portfolio is the numéraire. We are indebted to Profs. Steven Shreve and Julien Hugonnier for helpful discussions on these issues.

Example 9.1 A European call-option. Consider the contingent claim $Y = \big(X_1(T) - q\big)^+$: this is a European call-option with strike $q > 0$ on the first stock. Let us assume also that the interest-rate process $r(\cdot)$ is bounded away from zero, namely that $P[r(t) \geq r,\ \forall t \geq 0] = 1$ holds for some $r > 0$, and that the market $\mathcal{M}$ is weakly diverse on all time-horizons $T \in (0, \infty)$ sufficiently large. Then for the hedging price of this contingent claim, written now as a function $h(T)$ of the time-horizon, we have from (9.10), (9.13) and $E[L(T)] < 1$:

$$\begin{aligned} X_1(0) &> E[L(T)X_1(T)/B(T)] \geq E[L(T)(X_1(T)-q)^+/B(T)] = h(T) \\ &\geq \Big(E[L(T)X_1(T)/B(T)] - q \cdot E\Big(L(T) \cdot e^{-\int_0^T r(t)dt}\Big)\Big)^+ \\ &\geq \big(E[L(T)X_1(T)/B(T)] - qe^{-rT}E[L(T)]\big)^+ \\ &\geq \big(E[L(T)X_1(T)/B(T)] - qe^{-rT}\big)^+ \end{aligned}$$

because $L(\cdot)X_1(\cdot)/B(\cdot)$ is a supermartingale and a strict local martingale.

Therefore

$$0 \leq h(\infty) := \lim_{T\to\infty} h(T) = \lim_{T\to\infty} \downarrow E\left(\frac{L(T)X_1(T)}{B(T)}\right) < X_1(0): \tag{9.14}$$

the hedging price of the option is strictly less than the capitalization of the underlying stock at time $t = 0$, and tends to $h(\infty) \in [0, X_1(0))$ as the horizon increases without limit.

We claim that if $\mathcal{M}$ is uniformly weakly diverse over some $[T_0, \infty)$, then the limit in (9.14) is actually zero: a European call-option that can never be exercised has zero hedging price. Indeed, for every fixed $p \in (0,1)$ and $T \geq \frac{2\log n}{p\varepsilon\delta} \vee T_0$, the quantity

$$\begin{aligned} E\left(\frac{L(T)}{B(T)}X_1(T)\right) &\leq E\left(\frac{L(T)}{B(T)}Z^\mu(T)\right) \\ &\leq E\left(\frac{L(T)}{B(T)}Z^{\pi^{(p)}}(T)\right) \cdot n^{\frac{1-p}{p}} e^{-\varepsilon\delta(1-p)T/2} \end{aligned}$$

is dominated by $Z(0)n^{\frac{1-p}{p}} \cdot e^{-\varepsilon\delta(1-p)T/2}$, from (3.1), (11.4) and the supermartingale property of $L(\cdot)Z^{\pi^{(p)}}(\cdot)/B(\cdot)$. Letting $T \to \infty$ as in (9.14), this leads to $h(\infty) = 0$.

Remark 9.4 Note the sharp difference between this case and the situation where an equivalent martingale measure exists on every finite time-horizon; namely, when both $L(\cdot)$ and $L(\cdot)X_1(\cdot)/B(\cdot)$ are martingales. Then $E[L(T)X_1(T)/B(T)] = X_1(0)$ holds for all $T \in (0, \infty)$, and $h(\infty) = X_1(0)$: as the time-horizon increases without limit, the hedging price of the call-option approaches the current stock value (Karatzas and Shreve 1998, p. 62).

Example 9.2 Put-call parity. Suppose that $\Xi_1(\cdot), \Xi_2(\cdot)$ are positive, continuous and adapted processes, representing the values of two different assets in a market $\mathcal{M}$ with $r(\cdot) \equiv 0$. Let us set $Y_1 := \Big(\Xi_1(T) - \Xi_2(T)\Big)^+$ and $Y_2 := \Big(\Xi_2(T) -$

$\Xi_1(T)\Big)^+$; then from (9.13) the quantity $h_j = E[L(T)Y_j]$ is the hedging price at time $t = 0$ of a contract that offers its holder the right, but not the obligation, to exchange asset 2 for asset 1 with $j = 1$ (resp., asset 1 for asset 2 with $j = 2$) at time $t = T$. We have clearly

$$h_1 - h_2 = E[L(T)(\Xi_1(T) - \Xi_2(T))],$$

and say that the two assets are in put-call parity if $h_1 - h_2 = \Xi_1(0) - \Xi_2(0)$. This will be the case when both $L(\cdot)\Xi_1(\cdot)$, $L(\cdot)\Xi_2(\cdot)$ are martingales. (For instance, whenever (4.6) is valid we can take $\Xi_j(\cdot) \equiv X_i(\cdot)$ or $\Xi_j(\cdot) \equiv Z^\pi(\cdot)$ for any $i = 1, \cdots, n$, $j = 1, 2$ and any extended portfolio $\pi(\cdot)$; then put-call parity holds as in Karatzas and Shreve (1998, p. 50.)

It is easy to see that *put-call parity need not hold if $\mathcal{M}$ is weakly diverse:* for instance, take $\Xi_1(\cdot) \equiv Z^\mu(\cdot)$, $\Xi_2(\cdot) \equiv Z^{\widehat{\pi}}(\cdot)$ with $Z^\mu(0) = Z^{\widehat{\pi}}(0)$ in the notation of (3.1) and (8.11), and observe from (8.10) that $h_1 - h_2 = E[L(T)(Z^\mu(T) - Z^{\widehat{\pi}}(T))] > 0 = Z^\mu(0) - Z^{\widehat{\pi}}(0)$. (A similar observation appears in Lowenstein and Willard 2000.)

10 Concluding remarks

We have presented examples of diverse and weakly diverse market models posited in Fernholz (1999, 2002), and shown that the diversity-weighted portfolio of (4.4) represents an arbitrage relative to a weakly-diverse market over sufficiently long time-horizons. We have also shown that weakly-diverse markets are themselves arbitrages relative to suitable extended portfolios, over arbitratry time-horizons. In particular, no equivalent martingale measure can exist for such markets. But we have also shown that, even in diverse markets, this does not interfere with the development of option pricing; quite the contrary, one is led to more realistic hedging-prices for warrants over exceedingly long time-horizons. Similar treatments are possible for solving utility maximization problems along the lines of Karatzas et al. (1991), for showing that diversity is compatible with economic equilibrium considerations as in Chapt. 4 of Karatzas and Shreve (1998), and for treating general semimartingale market models (see Kardaras 2004). It would be of interest to determine the optimal hedging strategy $\widehat{\varphi}(\cdot)$ under suitable (e.g., Markovian) structure conditions, and to treat in the framework of Sect. 9 the hedging of American options.

Appendix: Proofs of selected results

Proof of (5.10)–(5.12) With $e_i = (0, \cdots, 0, 1, 0, \cdots, 0)'$ the i^{th} unit vector in $I\!R^n$, we have $\tau^\pi_{ii}(t) = \big(\pi(t) - e_i\big)' a(t)\big(\pi(t) - e_i\big) \geq \varepsilon||\pi(t) - e_i||^2 = \varepsilon\left[\big(1 - \pi_i(t)\big)^2 + \sum_{j\neq i} \pi_j^2(t)\right] \geq \varepsilon\big(1 - \pi_i(t)\big)^2$ from (5.3) and (2.3). Back into

(5.4), this gives

$$\begin{aligned}\gamma_*^\pi(t) &\geq \frac{\varepsilon}{2}\cdot\sum_{i=1}^n \pi_i(t)\left[\big(1-\pi_i(t)\big)^2+\sum_{j\neq i}\pi_j^2(t)\right]\\ &= \frac{\varepsilon}{2}\cdot\left[\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)^2+\sum_{j=1}^n \pi_j^2(t)\big(1-\pi_j(t)\big)\right]\\ &= \frac{\varepsilon}{2}\cdot\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big) \geq \frac{\varepsilon}{2}\big(1-\pi_{(1)}(t)\big).\end{aligned}$$

Similarly, we get $\tau_{ii}^\pi(t) \leq M\left[\big(1-\pi_i(t)\big)^2+\sum_{j\neq i}\pi_j^2(t)\right] \leq M\big(1-\pi_i(t)\big)\big(2-\pi_i(t)\big)$ as claimed in (5.10), and this leads to (5.11) and to

$$\begin{aligned}\gamma_*^\pi(t) &\leq \frac{M}{2}\cdot\sum_{i=1}^n \pi_i(t)\big(1-\pi_i(t)\big)\\ &= \frac{M}{2}\cdot\left[\pi_{(1)}(t)\big(1-\pi_{(1)}(t)\big)+\sum_{k=2}^n \pi_{(k)}(t)\big(1-\pi_{(k)}(t)\big)\right]\\ &\leq \frac{M}{2}\cdot\left[\big(1-\pi_{(1)}(t)\big)+\sum_{k=2}^n \pi_{(k)}(t)\right] = M\big(1-\pi_{(1)}(t)\big).\end{aligned}$$

Proof of (4.5) Let us start by introducing the function $D(x) := \left(\sum_{i=1}^n x_i^p\right)^{1/p}$, which we shall interpret as a "measure of diversity". An application of Itô's rule to the process $\{D(\mu(t)),\ 0 \leq t < \infty\}$ leads after some computation to the expression

$$\log\left(\frac{Z^{\pi^{(p)}}(T)}{Z^\mu(T)}\right) = \log\left(\frac{D(\mu(T))}{D(\mu(0))}\right) + (1-p)\int_0^T \gamma_*^{\pi^{(p)}}(t)dt, \qquad 0 \leq T < \infty \tag{11.1}$$

for the value-process $Z_{\pi^{(p)}}(\cdot)$ of the diversity-weighted portfolio $\pi^{(p)}(\cdot)$ of (4.4). Useful in the computation (11.1) is the numéraire-invariance property (5.5).

Suppose that the market is weakly diverse on the finite time-horizon $[0, T]$, namely, that $\int_0^T \Big(1-\mu_{(1)}(t)\Big)dt > \delta T$ holds almost surely, for some $0 < \delta < 1$. We have then $1 = \sum_{i=1}^n \mu_i(t) \leq \sum_{i=1}^n \big(\mu_i(t)\big)^p = \Big(D(\mu(t))\Big)^p \leq n^{1-p}$ (minimum diversity occurs when the entire market is concentrated in one stock, and maximum diversity when all stocks have the same capitalization), so that

$$\log\left(\frac{D(\mu(T))}{D(\mu(0))}\right) \geq -\frac{1-p}{p}\cdot\log n. \tag{11.2}$$

This provides, in particular, the lower bound $Z^{\pi^{(p)}}(\cdot)/Z^\mu(\cdot) \geq n^{-(1-p)/p}$. On the other hand, we have already remarked in Sect. 4 that the largest weight of the portfolio $\pi^{(p)}(\cdot)$ in (4.4) does not exceed the maximum weight of the market portfolio, namely

$$\pi^{(p)}_{(1)}(t) := \max_{1\le i\le n} \pi^{(p)}_i(t) = \frac{\big(\mu_{(1)}(t)\big)^p}{\sum_{k=1}^n \big(\mu_{(k)}(t)\big)^p} \le \mu_{(1)}(t) \tag{11.3}$$

(the reverse inequality holds for the smallest weights, namely $\pi^{(p)}_{(n)}(t) := \min_{1\le i\le n} \pi^{(p)}_i(t) \ge \mu_{(n)}(t)$). From (5.12) and (11.3) we see that the aassumption (4.2) of weak diversity implies

$$\int_0^T \gamma_*^{\pi^{(p)}}(t)dt \ge \frac{\varepsilon}{2}\cdot \int_0^T \Big(1-\mu_{(1)}(t)\Big)dt > \frac{\varepsilon}{2}\cdot \delta T, \qquad \text{a.s.}$$

and in conjunction with (11.2) this lead to (4.5) via

$$\log\left(\frac{Z^{\pi^{(p)}}(T)}{Z^{\mu}(T)}\right) > (1-p)\left[\frac{\varepsilon T}{2}\cdot\delta - \frac{1}{p}\cdot \log n\right]. \tag{11.4}$$

Now if $\mathcal{M}$ is uniformly weakly diverse over $[T_*,\infty)$, then (11.4) gives the long-term comparison $\mathcal{L}^{\pi^{(p)},\mu} = \underline{\lim}_{T\to\infty} \frac{1}{T}\log(Z^{\pi^{(p)}}(T)/Z^{\mu}(T)) \ge (1-p)\frac{\varepsilon\delta}{2} > 0$, a.s.

Proof that the martingale property of $L(\cdot)$ (valid under condition (4.6)) proscribes (2.10). Suppose that the exponential process $\{L(t); 0\le t\le T\}$ of (9.5) is a $P-$martingale; then $\{\widehat{W}(t); 0\le t\le T\}$ of (9.2) is Brownian motion under the equivalent probability measure $Q_T(A) = E[L(T)\cdot 1_A]$ on $\mathcal{F}(T)$, by the Girsanov theorem. For instance, this will be the case under the Novikov condition (4.6); cf. Theorem 3.5.1 and Proposition 3.5.12 in Karatzas and Shreve (1991). For any extended portfolio $\pi(\cdot)$ we have

$$d(Z^{z,\pi}(t)/B(t)) = (Z^{z,\pi}(t)/B(t))\cdot \sum_{i=1}^n\sum_{\nu=1}^m \pi_i(t)\sigma_{i\nu}(t)d\widehat{W}_\nu(t), \quad Z^{z,\pi}(0)=z>0$$

from (9.4) and the discussion following it; this shows that under Q_T the process $Z^{z,\pi}(\cdot)/B(\cdot)$ is then a martingale with moments of all orders (in particular, square-integrable). If $\rho(\cdot)$ is another extended portfolio, the difference $H(\cdot) := (Z^{z,\pi}(\cdot) - Z^{z,\rho}(\cdot))/B(\cdot)$ is again a (square-integrable) martingale with $H(0)=0$, therefore $E^{Q_T}[H(T)] = 0$. But if $H(T)\ge 0$ holds a.s. (with respect to P, or equivalently with respect to Q_T), then this gives $H(T) = 0$ a.s. and rules out the second requirement $P[H(T)>0]>0$ of (2.10).

Proof of (9.10). Suppose that the processes $L(\cdot)X_i(\cdot)/B(\cdot)$ for $i=1,\cdots,n$ are all martingales; then so is their sum, the process $\widetilde{Z}(\cdot) := L(\cdot)Z^{\mu}(\cdot)/B(\cdot)$ with $Z^{\mu}(\cdot) := \sum_{i=1}^n X_i(\cdot)$ as in (3.1). With $z=1$, $\varphi(\cdot)\equiv Z^{\mu}(\cdot)\mu(\cdot)$ and $\vartheta^{\mu}(t) := \sigma'(t)\mu(t) - \vartheta(t)$, the Eq. (9.9) takes the form $d\widetilde{Z}^{\mu}(t) = \widetilde{Z}^{\mu}(t)(\vartheta^{\mu}(t))'dW(t)$ or equivalently

$$\widetilde{Z}^{\mu}(t) = \exp\left(\int_0^t (\vartheta^{\mu}(s))'dW(s) - \frac{1}{2}\int_0^t ||\vartheta^{\mu}(s)||^2 ds\right), \tag{11.5}$$

and we get

$$\frac{1}{\widetilde{Z}^{\mu}(t)} = \exp\left(-\int_0^t (\vartheta^{\mu}(s))' d\widetilde{W}(s) - \frac{1}{2}\int_0^t ||\vartheta^{\mu}(s)||^2 ds\right),$$
$$\text{where} \quad \widetilde{W}(\cdot) := W(\cdot) - \int_0^{\cdot} \vartheta^{\mu}(s) ds.$$

Now on any given finite horizon $[0,T]$, this process $\widetilde{W}(\cdot)$ is Brownian motion under the equivalent probability measure $\widetilde{P}_T(A) := E[\widetilde{Z}^{\mu}(T) \cdot 1_A]$ on $\mathcal{F}(T)$, and Itô's rule gives

$$d\left(\frac{Z^{\pi}(t)}{Z^{\mu}(t)}\right) = \left(\frac{Z^{\pi}(t)}{Z^{\mu}(t)}\right) \cdot \sum_{i=1}^{n}\sum_{\nu=1}^{m} (\pi_i(t) - \mu_i(t))\sigma_{i\nu}(t) d\widetilde{W}_{\nu}(t) \tag{11.6}$$

for an arbitrary extended portfolio $\pi(\cdot)$. From (2.3) we see that, for any such $\pi(\cdot)$, the ratio $Z^{\pi}(\cdot)/Z^{\mu}(\cdot)$ is a martingale under $\widetilde{P}_T$; in particular, $E^{\widetilde{P}_T}[Z^{\pi}(T)/Z^{\mu}(T)] = 1$. But if $\pi(\cdot)$ satisfies $P[Z^{\pi}(T) \geq Z^{\mu}(T)] = 1$, we must have also $\widetilde{P}_T[Z^{\pi}(T)/Z^{\mu}(T) \geq 1] = 1$; in conjunction with $E^{\widetilde{P}_T}[Z^{\pi}(T)/Z^{\mu}(T)] = 1$, this leads to $\widetilde{P}_T[Z^{\pi}(T) = Z^{\mu}(T)] = 1$, or equivalently $Z^{\pi}(T) = Z^{\mu}(T)$ a.s. P, contradicting (4.5). Thus the process

$$\widetilde{X}_j(t) = \exp\left(\int_0^t (\vartheta^{(j)}(s))' dW(s) - \frac{1}{2}\int_0^t ||\vartheta^{(j)}(s)||^2 ds\right), \quad 0 \leq t < \infty \tag{11.7}$$

of (9.8) is a strict local martingale, for some (at least one) $j \in \{1, \cdots, n\}$; we have set $\vartheta_{\nu}^{(k)}(t) := \sigma_{k\nu}(t) - \vartheta_{\nu}(t),\ \nu = 1, \cdots, n,$ for any $k \in \{1, \cdots, n\}$.

Suppose now that (9.10) fails, i.e., that $\widetilde{X}_i(\cdot)$ is a martingale for some $i \neq j$. Then for any $T \in (0, \infty)$ the measure $P_T^{(i)}(A) := E[\widetilde{X}_i(T) \cdot 1_A]$ is a probability on $\mathcal{F}(T)$, under which the process $\widetilde{W}^{(i)}(t) := W(t) - \int_0^t \vartheta^{(i)}(s) ds,\ 0 \leq t \leq T$ is standard $I\!R^n-$valued Brownian motion. By analogy with (11.5)–(11.7) we have now

$$\frac{1}{\widetilde{X}_i(t)} = \exp\left(-\int_0^t (\vartheta^{(i)}(s))' d\widetilde{W}^{(i)}(s) - \frac{1}{2}\int_0^t ||\vartheta^{(i)}(s)||^2 ds\right),$$

and

$$d\left(\frac{X_j(t)}{X_i(t)}\right) = \left(\frac{X_j(t)}{X_i(t)}\right) \cdot \sum_{\nu=1}^{m} (\sigma_{j\nu}(t) - \sigma_{i\nu}(t)) d\widetilde{W}_{\nu}^{(i)}(t).$$

Thus, thanks to condition (2.3), the process $X_j(\cdot)/X_i(\cdot)$ is a $P_T^{(i)}-$martingale on $[0,T]$, with moments of all orders. In particular,

$$\frac{X_j(0)}{X_i(0)} = E^{P_T^{(i)}}\left[\frac{X_j(T)}{X_i(T)}\right] = E\left[\frac{L(T)X_i(T)}{B(T)X_i(0)} \cdot \frac{X_j(T)}{X_i(T)}\right],$$

which contradicts $E[L(T)X_j(T)/B(T)] < X_j(0)$ and thus the strict local martingale property of $L(\cdot)X_j(\cdot)/B(\cdot)$ under P.

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