

Market viability via absence of arbitrage of the first kind

Constantinos Kardaras

Received: 7 January 2010 / Accepted: 21 December 2010 / Published online: 23 February 2012
© Springer-Verlag 2012

Abstract It is shown that, in a semimartingale financial market model, there is equivalence between absence of *arbitrage of the first kind* (a weak viability condition) and the existence of a strictly positive process that acts as a local martingale deflator on nonnegative wealth processes.

Keywords Arbitrage of the first kind · Cheap thrills · Fundamental theorem of asset pricing · Equivalent local martingale deflators · Semimartingales · Predictable characteristics

Mathematics Subject Classification (2010) 60G44 · 60H99 · 91B28 · 91B70

JEL Classification G10

1 Introduction

A ubiquitous market assumption in the literature of stochastic finance theory is postulating the existence of an equivalent local martingale measure (ELMM). The latter refers to a probability measure \mathbb{Q} , equivalent to the “real-world” probability \mathbb{P} , with the property that all discounted nonnegative wealth processes are local \mathbb{Q} -martingales. In view of the fundamental theorem of asset pricing (FTAP), it is quite clear why such an assumption is made from the outset: existence of an ELMM is intimately connected to market viability; in fact, it is equivalent to the economically sound “no free lunch with vanishing risk” (NFLVR) condition—see for example [4] and [5] for a complete treatment on the topic.

C. Kardaras (✉)

Mathematics and Statistics Department, Boston University, 111 Cummington Street, Boston, MA 02215, USA
e-mail: kardaras@bu.edu

Stipulating the existence of an ELMM seems unavoidable in order to maintain market viability. However, in recent publications there has been considerable interest in models where an ELMM might fail to exist. These have appeared, for instance,

- in the context of stochastic portfolio theory, for which the survey [6] is a good introduction;
- from the financial modelling perspective, an example of which is the *benchmark approach* of [20];
- in a financial equilibrium setting, both for infinite horizon settings (see [8]), as well as finite horizon models with credit constraints on economic agents (see [18, 19]).

The common assumption that the previous approaches share is postulating the existence of an equivalent local martingale deflator (ELMD), that is, a strictly positive process that makes all discounted nonnegative wealth processes, when multiplied by it, local martingales. (An ELMD was called a *strict martingale density* in [23]; we opt here to call it ELMD as it immediately connects with the notion of an ELMM.) An ELMD is a strictly positive local martingale, but not necessarily a martingale; therefore, it cannot always be used as a density process to produce an ELMM.

While models where an ELMM might fail to exist are now being extensively studied, a result that would justify their applicability along the lines of the FTAP has not yet appeared in the literature. In this work, the aforementioned issue is tackled. A precise economic condition of market viability is given using the concept of *arbitrage of the first kind*, which has first appeared under this appellation in [10]; see also [13] in the context of large financial markets, as well as [18], where arbitrage of the first kind is called a *cheap thrill*. Absence of arbitrage of the first kind in the market, which we shall abbreviate as condition NA_1 , is close in spirit, but strictly weaker, than the condition NFLVR; in fact, it is exactly equivalent to the condition “no unbounded profit with bounded risk” (NUPBR) that appeared in [16]. The main result of the present paper precisely states that in a one-dimensional semimartingale market model, there is equivalence between condition NA_1 and the existence of an ELMD.

In the literature concerning discrete-time models, there have appeared two ways of providing a proof of the FTAP. The first one is the approach of [3] (initiated in [9]), which utilizes convex separation via functional-analytic arguments. The alternative, presented in [21], uses the economic idea that the marginal utility evaluated at the optimal terminal wealth of an economic agent, when properly scaled, defines the density of an equivalent martingale measure. The former approach has been adapted with extreme success to continuous-time models in [4] and [5]. The present work can be seen as a counterpart of the latter approach in continuous-time markets—here, the utility involved is logarithmic (under a suitable change of probability), and makes the reciprocal of the log-optimizer an ELMD. Interestingly enough, in continuous-time models the two approaches do not give rise to the same result; the present approach weakens the equivalent conditions of the classical FTAP in [4], both from the mathematical and the economic side. Note that the main result of this paper can also be seen as an intermediate step in proving the general version of the FTAP as it is presented in [4]. In fact, this task is taken up in [17].

The structure of the paper is simple. In Sect. 2, the market is introduced, arbitrage of the first kind is defined and the main result is stated. Its somewhat lengthy and technical proof is deferred to Sect. 3.

2 Absence of arbitrage of the first kind and equivalent local martingale deflators

2.1 Probabilistic remarks

All stochastic processes in the sequel are defined on a *filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}(t))_{t \in \mathbb{R}_+}, \mathbb{P})$. All relationships between random variables are understood in the \mathbb{P} -a.s. sense. The filtration $(\mathcal{F}(t))_{t \in \mathbb{R}_+}$ is right-continuous. We assume the existence of a finite financial planning horizon T , where T is a *finite* stopping time. All processes will be assumed to be constant, and equal to their value they have at T , after time T . Without affecting the generality of the discussion, it will be assumed throughout that $\mathcal{F}(0)$ is trivial modulo \mathbb{P} and that $\mathcal{F}(T) = \mathcal{F}$.

2.2 Investment

Let S be a *real-valued semimartingale*, denoting the *discounted*, with respect to some baseline security, price process of a financial security. Starting with capital $x \in \mathbb{R}$, and investing according to some predictable and S -integrable strategy ϑ , an economic agent's discounted wealth process is

$$X^{x, \vartheta} := x + \int_0^{\cdot} \vartheta(t) dS(t). \quad (2.1)$$

When modelling frictionless trading, credit constraints have to be imposed on investment in order to avoid *doubling strategies*. Define then \mathcal{X} to be the set of all nonnegative wealth processes, i.e., all $X^{x, \vartheta}$ in the notation of (2.1) such that $X^{x, \vartheta} \geq 0$.

2.3 Equivalent local martingale deflators

An *equivalent local martingale deflator* (ELMD) is a nonnegative process Z with $Z(0) = 1$ and $Z(T) > 0$ such that ZX is a local martingale for all $X \in \mathcal{X}$. Since $1 \equiv X^{1,0} \in \mathcal{X}$, an ELMD is in particular a strictly positive local martingale.

2.4 Arbitrage of the first kind

An $\mathcal{F}(T)$ -measurable random variable ξ will be called an *arbitrage of the first kind* if $\mathbb{P}[\xi \geq 0] = 1$, $\mathbb{P}[\xi > 0] > 0$, and for all $x > 0$, there exists $X^{x, \vartheta} \in \mathcal{X}$ (for some ϑ which may depend on x), such that $X^{x, \vartheta}(T) \geq \xi$. If there exists no arbitrage of the first kind in the market, we shall say that condition NA₁ holds.

It is straightforward to see that condition NA₁ is weaker than condition NFLVR of [4]. (The reader should consult Remark 2.4 later in the text for a disambiguation of a possible confusion that could arise from such a claim, especially in the context of large financial markets.) Actually, using a combination of Lemma A1.1 in [4] and Lemma 2.3 in [2], it is shown in [17, Proposition 1.2] that condition NA₁ is equivalent to the requirement that the set $\{X(T) \mid X \in \mathcal{X}$ with $X_0 = 1\}$ is bounded in probability. The latter condition has been coined BK in [12] and NUPBR in [16].

2.5 The main result

The next result can be seen as a weak version of the FTAP. Though simple to state, its proof is quite technical and is given in Sect. 3.

Theorem 2.1 *Condition NA_1 is equivalent to the existence of at least one ELMD.*

Remark 2.2 In [17], which is in a certain sense a sequel to this paper, it is argued that although an ELMD does not generate a probability measure, its local martingale structure allows one to define a *finitely additive* probability that is *locally countably additive* and *weakly equivalent* to \mathbb{P} , and further makes discounted asset price processes behave like “local martingales”. More precisely, Theorem 2.1 can be reformulated to state that condition NA_1 is valid if and only if there exists $Q : \mathcal{F} \rightarrow [0, 1]$ and a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times with $\lim_{n \rightarrow \infty} \mathbb{P}[\tau_n = T] = 1$ such that:

- $Q[\emptyset] = 0$, $Q[\Omega] = 1$, and Q is (finitely) additive: $Q[A \cup B] = Q[A] + Q[B]$ whenever $A \in \mathcal{F}$ and $B \in \mathcal{F}$ satisfy $A \cap B = \emptyset$;
- for $A \in \mathcal{F}$, $\mathbb{P}[A] = 0$ implies $Q[A] = 0$;
- when restricted on \mathcal{F}_{τ_n} , Q is countably additive and equivalent to \mathbb{P} , for all $n \in \mathbb{N}$.
- $\int_{\Omega} X_{\tau^n \wedge \tau} dQ = X_0$ holds for all $X \in \mathcal{X}$, $n \in \mathbb{N}$ and all stopping times τ .

Using this reformulation, Theorem 2.1 bears more resemblance to the FTAP of [4]. In fact, as already mentioned in the introduction, in [17] Theorem 2.1 is used as an intermediate step in proving the FTAP in [4].

Remark 2.3 Theorem 2.1 is stated for one-dimensional semimartingales S , as even for this “simple” case the proof is quite technical and requires taking care of many different issues, as the reader will appreciate in Sect. 3 below. There is no doubt that the result is still valid for the multidimensional semimartingale case, albeit its proof is expected to be significantly more involved.

Remark 2.4 Condition NA_1 has been extensively used in the setting of large financial markets—see for example [13]. In that classical context, one starts with a sequence of “small” financial markets, each individually possessing an ELMM, and studies viability of a limiting “large” financial market. In the “stationary” case, meaning that all markets are identical, it trivially follows that the NA_1 condition is equivalent to NFLVR, since the existence of an ELMM is assumed from the outset. In contrast, in the setting of the present paper, one does not impose any a priori normative condition on the market structure—this would anyway render the statement of Theorem 2.1 almost tautological. As a final remark, note that in the literature there have also been treatments of large financial markets where one only stipulates the existence of an equivalent *supermartingale* deflator for the sequence of approximating markets—see for example [22].

3 The proof of Theorem 2.1

3.1 Proving Theorem 2.1 with the help of an auxiliary result

The proof of one implication of Theorem 2.1 is easy and somewhat classic, but will be presented here anyhow for completeness. Start by assuming the existence of an ELMD Z and pick any sequence $(X_k)_{k \in \mathbb{N}}$ of wealth processes in \mathcal{X} such that $\lim_{k \rightarrow \infty} X_k(0) = 0$ as well as $X_k(T) \geq \xi$ for some nonnegative random variable ξ . Since ZX_k is a nonnegative local martingale, thus a \mathbb{P} -supermartingale,

$$\mathbb{E}[Z(T)\xi] \leq \mathbb{E}[Z(T)X_k(T)] \leq Z(0)X_k(0) = X_k(0)$$

holds for all $k \in \mathbb{N}$. Therefore, $\mathbb{E}[Z(T)\xi] \leq 0$. Since $\mathbb{P}[Z(T) > 0, \xi \geq 0] = 1$, $\mathbb{E}[Z(T)\xi] \leq 0$ holds only if $\mathbb{P}[\xi = 0] = 1$. Therefore, ξ cannot be an arbitrage of the first kind, and condition NA_1 holds.

It remains to prove the other implication, which is considerably harder. Define

$$\mathcal{X}_{++} := \{X \in \mathcal{X} \mid X > 0 \text{ and } X_- > 0\}.$$

Since condition NA_1 is equivalent to condition NUPBR of [16], the general results of the latter paper imply that condition NA_1 is equivalent to the existence of $\hat{X} \in \mathcal{X}_{++}$ with $\hat{X}(0) = 1$ such that, with $Z := 1/\hat{X}$, ZX is a supermartingale for all $X \in \mathcal{X}_{++}$. (Note that the results of [16] have been established when $S \in \mathcal{X}_{++}$; however, this condition is unnecessary. At any rate, in the present paper we give a full treatment instead of depending on results from [16].) Unfortunately, when jumps are present in S , these last supermartingales might fail to be local martingales. In order to achieve our goal, we shall have to slightly alter the original probability using the predictable characteristics of S . (The idea of how to perform such a change of probability is already present in [12] and [7].) In Sect. 3.2 below, we shall establish the following result, certainly interesting in its own right. Before stating it, recall that for a signed measure μ on $(\mathcal{Q}, \mathcal{F})$, its *total variation* norm is defined as $|\mu|_{TV} := \sup_{A \in \mathcal{F}} |\mu[A]|$.

Theorem 3.1 *Assume that condition NA_1 holds. Then for any $\epsilon > 0$, there exists a probability $\tilde{\mathbb{P}} = \tilde{\mathbb{P}}(\epsilon)$ with the following properties:*

1. $\tilde{\mathbb{P}}$ is equivalent to \mathbb{P} on $\mathcal{F}(T)$.
2. $|\tilde{\mathbb{P}} - \mathbb{P}|_{TV} \leq \epsilon$.
3. *There exists $\tilde{X} \in \mathcal{X}_{++}$ with $\tilde{X}(0) = 1$ such that X/\tilde{X} is a local $\tilde{\mathbb{P}}$ -martingale for all $X \in \mathcal{X}$.*

To see how Theorem 3.1 completes the proof of Theorem 2.1, assume that condition NA_1 holds, as well as the statement of Theorem 3.1. Define the process Z via $Z_t := (1/\tilde{X}(t))(\tilde{\mathbb{P}}/\mathbb{P})|_{\mathcal{F}(t)}$ for $t \in \mathbb{R}_+$, where $(\tilde{\mathbb{P}}/\mathbb{P})|_{\mathcal{F}(t)}$ denotes the Radon–Nikodým derivative of $\tilde{\mathbb{P}}$ with respect to \mathbb{P} when the two probabilities are restricted on the sigma-algebra $\mathcal{F}(t)$. Then Theorem 3.1(1) implies that $Z(0) = 1$ and $Z(T) > 0$, and the fact that ZX is a local martingale for all $X \in \mathcal{X}$ follows by Theorem 3.1(3).

3.2 The proof of Theorem 3.1

In the course of the proof, results from the general theory of stochastic processes from [11] are used. There are ideas from [16] that are utilized throughout the proof; as the latter paper is long and technical, and in an effort to be as self-contained as possible, we are providing full arguments whenever possible. In fact, there is only one result from [16] whose statement will just be assumed; this happens at the end of Sect. 3.2.4.

3.2.1 Predictable characteristics

In order to prove Theorem 3.1, we can assume without loss of generality that S is a special semimartingale under \mathbb{P} . Indeed, if this is not the case, we can change the original probability \mathbb{P} into another equivalent $\bar{\mathbb{P}}$ using the Radon–Nikodým density:

$$\frac{d\bar{\mathbb{P}}}{d\mathbb{P}} := \frac{1}{\mathbb{E}[(1 + \gamma \sup_{t \in \mathbb{R}_+} |S(t)|)]^{-1}} \left(1 + \gamma \sup_{t \in \mathbb{R}_+} |S(t)|\right)^{-1},$$

where $\gamma > 0$ is small enough so that $|\bar{\mathbb{P}} - \mathbb{P}|_{\text{TV}} \leq \epsilon/2$. Then $\bar{\mathbb{E}}[\sup_{t \in \mathbb{R}_+} |S(t)|] < \infty$, where $\bar{\mathbb{E}}$ denotes expectation under $\bar{\mathbb{P}}$; in particular, S is a special semimartingale under $\bar{\mathbb{P}}$. Then the validity of Theorem 3.1 can be shown for $\bar{\mathbb{P}}$ with $\epsilon/2$ replacing ϵ .

Now, assuming that S is a special semimartingale under \mathbb{P} , write its *canonical decomposition* $S = S_0 + A + S^c + \int_{(0, \cdot] \times \mathbb{R}} x(\mu[dt, dx] - \nu[dt, dx])$. Here, A is *predictable and of finite variation*, S^c is a local martingale with *continuous* paths and $\int_{(0, \cdot] \times \mathbb{R}} x(\mu[dt, dx] - \nu[dt, dx])$ is a *purely discontinuous* local martingale. As usual, μ is the *jump measure* of S defined via $\mu(D) := \sum_{t \in \mathbb{R}_+} \mathbb{I}_D(t, \Delta S(t)) \mathbb{I}_{\mathbb{R} \setminus \{0\}}(t)$ for $D \subseteq \mathbb{R}_+ \times \mathbb{R}$, and ν is the *predictable compensator* of the measure μ . Since S is a special semimartingale, we have $\int_{\mathbb{R}_+ \times \mathbb{R}} (|x| \wedge |x|^2) \nu[dt, dx] < \infty$. We introduce the *quadratic covariation process* $C := [S^c, S^c]$ of S^c and define the predictable nondecreasing scalar process

$$G := C + \int_{(0, \cdot]} |dA(t)| + \int_{(0, \cdot] \times \mathbb{R}} (|x| \wedge |x|^2) \nu[dt, dx].$$

All three processes A , C , and ν are absolutely continuous with respect to G . Therefore, we can write

$$\begin{aligned} A &= \int_{(0, \cdot]} a(t) dG(t), & C &= \int_{(0, \cdot]} c(t) dG(t), \\ \nu[(0, \cdot] \times E] &= \int_{(0, \cdot]} \kappa(t)[E] dG(t), \end{aligned}$$

where a , c and κ are predictable, a is a scalar process, c a nonnegative scalar process, κ a process with values in the set of measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that do not charge $\{0\}$ and integrate the function $\mathbb{R} \ni x \mapsto |x| \wedge |x|^2$, and $E \in \mathcal{B}(\mathbb{R})$.

Condition NA₁ enforces some restrictions on the triplet of predictable characteristics of S . The next result is a consequence of [16, Theorem 3.15(2)], but we provide the quick argument for completeness.

Lemma 3.2 *Assume condition NA₁ in the market. Then, with $\Lambda := \Lambda_+ \cup \Lambda_-$, where*

$$\Lambda_+ := \left\{ \kappa[(-\infty, 0)] = 0, c = 0, a > \int_{(0, \infty)} x \kappa[\mathrm{d}x] \right\},$$

$$\Lambda_- := \left\{ \kappa[(0, \infty)] = 0, c = 0, a < \int_{(-\infty, 0)} x \kappa[\mathrm{d}x] \right\},$$

the predictable set Λ is $(\mathbb{P} \otimes G)$ -null. (In particular, $\{\kappa[\mathbb{R}] = 0, c = 0, a \neq 0\}$ is $(\mathbb{P} \otimes G)$ -null.)

Proof Define $\vartheta := \mathbb{I}_{\Lambda_+} - \mathbb{I}_{\Lambda_-}$. Then it is straightforward to see that

$$X^{0, \vartheta} = \int_{(0, \cdot]} \mathbb{I}_{\Lambda}(t) \left| a(t) - \int_{\mathbb{R}} x \kappa(t)[\mathrm{d}x] \right| \mathrm{d}G(t) + \sum_{t \in (0, \cdot]} \mathbb{I}_{\Lambda}(t) |\Delta S(t)|,$$

where we observe that the integral $\int_{\mathbb{R}} x \kappa[\mathrm{d}x]$ is always well defined on Λ . It is clear that $X^{0, \vartheta}$ is nondecreasing, i.e., $X^{0, \vartheta} \in \mathcal{X}$. Furthermore, if Λ fails to be $(\mathbb{P} \otimes G)$ -null, then $\mathbb{P}[X^{0, \vartheta}(T) > 0] > 0$. Let $\xi := X^{0, \vartheta}(T)$; since $X^{x, \vartheta}(T) = x + \xi \geq \xi$ for all $x > 0$, ξ is an arbitrage of the first kind. Therefore, under condition NA₁, Λ has to be $(\mathbb{P} \otimes G)$ -null. \square

3.2.2 Changes of probability

In what follows, a *strictly positive predictable random field* will refer to a function $Y : \Omega \times \mathbb{R}_+ \times \mathbb{R} \mapsto (0, \infty)$ that is measurable with respect to the product of the predictable sigma-algebra on $\Omega \times \mathbb{R}_+$ with the Borel sigma-algebra on \mathbb{R} . For any strictly positive predictable random field Y , let ν^Y be the predictable random measure that has density Y with respect to ν ; in other words, for all $E \in \mathcal{B}(\mathbb{R})$,

$$\nu^Y[(0, \cdot] \times E] = \int_{(0, \cdot]} \kappa^Y(t)[E] \mathrm{d}G(t) = \int_{(0, \cdot]} \left(\int_E Y(t, x) \kappa(t)[\mathrm{d}x] \right) \mathrm{d}G(t). \quad (3.1)$$

For all $t \in \mathbb{R}_+$, $Y(t, \cdot)$ is the density of $\kappa^Y(t)$ with respect to $\kappa(t)$.

Define the $(0, \infty)$ -valued predictable process

$$\eta := \frac{\epsilon}{2|1 + G|^2}.$$

In the sequel, we shall only consider strictly positive predictable random fields Y such that the following properties are additionally identically satisfied:

$$(Y1) \quad \int_{\mathbb{R}} (|x| \wedge |x|^2) \kappa^Y[\mathrm{d}x] < \infty.$$

$$(Y2) \quad \int_{\mathbb{R}} |Y(x) - 1| \kappa[\mathrm{d}x] \leq \eta.$$

$$(Y3) \quad \kappa[\mathbb{R}] = \kappa^Y[\mathbb{R}].$$

(The dependence of processes on $(\omega, t) \in \Omega \times \mathbb{R}_+$ is usually suppressed from notations to ease the reading. Whenever appropriate from the context, and for clarification purposes, we shall sometimes write $Y(x)$ or $Y(t, x)$ for Y .)

Property (Y2) of Y implies the estimate

$$\begin{aligned} \int_{\mathbb{R}_+ \times \mathbb{R}} |Y(t, x) - 1| \nu[dt, dx] &= \int_{\mathbb{R}_+} \left(\int_{\mathbb{R}} |Y(t, x) - 1| \kappa(t) [dx] \right) dG(t) \\ &\leq \int_{\mathbb{R}_+} \eta(t) dG(t) = \frac{\epsilon}{2} \int_{\mathbb{R}_+} \frac{dG(t)}{|1 + G(t)|^2} \leq \frac{\epsilon}{2}. \quad (3.2) \end{aligned}$$

It follows that the process $M := \int_{(0, \cdot] \times \mathbb{R}} (Y(t, x) - 1) (\mu[dt, dx] - \nu[dt, dx])$ is a well-defined local martingale. Observe that for all $t \in \mathbb{R}_+$, we have

$$\begin{aligned} \Delta M(t) &= Y(t, \Delta S(t)) - 1 - \left(\int_{\mathbb{R}} (Y(t, x) - 1) \kappa [dx] \right) \Delta G(t) \\ &= Y(t, \Delta S(t)) - 1 > -1 \end{aligned}$$

because Y is strictly positive, and $\int_{\mathbb{R}} (Y(t, x) - 1) \kappa [dx] = \kappa^Y[\mathbb{R}] - \kappa[\mathbb{R}] = 0$ due to (Y3). With \mathcal{E} denoting the *stochastic exponential* operator, define

$$L := \mathcal{E}(M) = \mathcal{E} \left(\int_{(0, \cdot] \times \mathbb{R}} (Y(t, x) - 1) (\mu[dt, dx] - \nu[dt, dx]) \right).$$

Combining (3.2) with $\Delta M > -1$, a use of [14, Theorem 12] gives that L is a uniformly integrable martingale with $\mathbb{P}[L(T) > 0] = 1$. However, because the last paper may be hard to obtain, we provide a quick argument in the present special case. At the same time, we show that the probability defined by L satisfies requirement (2) of Theorem 3.1.

Lemma 3.3 *Let Y be a strictly positive random field such that (Y1)–(Y3) hold. With the above notation, we have $\mathbb{P}[L(T) > 0] = 1$ and $\mathbb{E}[\sup_{t \in \mathbb{R}_+} |L(t) - 1|] \leq \epsilon$. In particular, the recipe $d\mathbb{P}^Y/d\mathbb{P} = L(T)$ defines a probability \mathbb{P}^Y that is equivalent to \mathbb{P} on $\mathcal{F}(T)$ such that $|\mathbb{P}^Y - \mathbb{P}|_{\text{TV}} \leq \epsilon$.*

Proof Since $\Delta M > -1$ and M is a local martingale, $\mathbb{P}[L(T) > 0] = 1$ follows.

Let $H := \int_{(0, \cdot]} |Y(t, x) - 1| \mu[dt, dx]$ and $F := \int_{(0, \cdot]} |Y(t, x) - 1| \nu[dt, dx]$. The process F is the predictable compensator of H and we have $\mathbb{P}[F(\infty) \leq \epsilon/2] = 1$ in view of (3.2). In particular, M is a local martingale of finite variation. Using the fact that $L = 1 + \int_{(0, \cdot]} L(t-) dM(t)$, we obtain

$$\begin{aligned} \mathbb{E} \left[\sup_{t \in \mathbb{R}_+} |L(t) - 1| \right] &\leq \mathbb{E} \left[\int_{(0, \infty)} L(t-) dH(t) + \int_{(0, \infty)} L(t-) dF(t) \right] \\ &= 2\mathbb{E} \left[\int_{(0, \infty)} L(t-) dF(t) \right]. \end{aligned}$$

Furthermore, with $(\tau_n)_{n \in \mathbb{N}}$ being a localizing sequence for L , we have

$$\begin{aligned} \mathbb{E}\left[\int_{(0, \tau_n]} L(t-) dF(t)\right] &= \mathbb{E}[L(\tau_n)F(\tau_n)] - \mathbb{E}\left[\int_{(0, \tau_n]} F(t) dL(t)\right] \\ &\leq \frac{\epsilon}{2} \mathbb{E}[L(\tau_n)] \leq \frac{\epsilon}{2}. \end{aligned}$$

As the previous is valid for each $n \in \mathbb{N}$, $\mathbb{E}[\sup_{t \in \mathbb{R}_+} |L(t) - 1|] \leq \epsilon$ follows from a straightforward application of the monotone convergence theorem. In particular, $\mathbb{E}[\sup_{t \in \mathbb{R}_+} |L(t)|] < \infty$ which implies that L is a uniformly integrable martingale and, therefore, \mathbb{P}^Y is well defined and equivalent to \mathbb{P} on $\mathcal{F}(T)$. Furthermore, $|\mathbb{P}^Y - \mathbb{P}|_{\text{TV}} = \mathbb{E}[|L(T) - 1|] \leq \epsilon$, which completes the proof. \square

Consider the probability \mathbb{P}^Y of Lemma 3.3. According to Girsanov's theorem (Theorem III.3.24 of [11]), under the assumptions (Y1)–(Y3) on Y , S is still a special semimartingale under \mathbb{P}^Y with canonical decomposition

$$S = S_0 + A^Y + S^{c,Y} + \int_{(0, \cdot) \times \mathbb{R}} x(\mu[dt, dx] - \nu^Y[dt, dx]),$$

where the predictable compensator ν^Y of μ under \mathbb{P}^Y was defined previously in (3.1) and $A^Y = \int_{(0, \cdot)} a^Y(t) dG(t)$ with $a^Y := a + \int_{\mathbb{R}} x(Y(x) - 1) \kappa(dx)$. For the continuous local \mathbb{P}^Y -martingale part $S^{c,Y}$, we have $C^Y := [S^{c,Y}, S^{c,Y}] = [S^c, S^c] = C$, i.e., $C^Y = \int_{(0, \cdot)} c^Y(t) dG(t)$ with $c^Y = c$.

3.2.3 Relative rate of return

Remember that Y always denotes a strictly positive predictable random field satisfying (Y1)–(Y3) of Sect. 3.2.2. We aim at understanding what extra condition Y must satisfy for $\tilde{\mathbb{P}} := \mathbb{P}^Y$ to satisfy all the requirements of Theorem 3.1.

Define a pair of processes (ℓ, r) via

$$\begin{aligned} \ell &:= \inf\{p \in \mathbb{R} \mid \kappa[\{x \in \mathbb{R} \mid 1 + px < 0\}] = 0\}, \\ r &:= \sup\{p \in \mathbb{R} \mid \kappa[\{x \in \mathbb{R} \mid 1 + px < 0\}] = 0\}. \end{aligned}$$

(ℓ and r are mnemonics for “left” and “right”, respectively.) It is straightforward that $\ell \leq 0 \leq r$, as well as that both ℓ and r are predictable; for example, $\{\ell \leq p\} = \Omega \times \mathbb{R}_+$ if $p \in \mathbb{R}_+$, while

$$\{\ell \leq p\} = \bigcap_{n \in \mathbb{N}} \{\kappa[\{x \in \mathbb{R} \mid 1 + (p + 1/n)x < 0\}] = 0\} \quad \text{if } p \in \mathbb{R} \setminus \mathbb{R}_+,$$

so that $\{\ell \leq p\}$ is predictable. Of course, nothing changes in the definition of ℓ and r if we replace κ with κ^Y . Define $I := [\ell, r] \cap \mathbb{R}$. Note that if conv.supp denotes the convex hull of the support of a measure, we have $\text{conv.supp}(\kappa) = [-1/r, -1/\ell] \cap \mathbb{R}$.

For two I -valued predictable processes p and p' , define a predictable process

$$\text{rel}^Y(p \mid p') := (p - p') \left(a^Y - p' c^Y - \int_{\mathbb{R}} \frac{p' |x|^2}{1 + p' x} \kappa^Y [dx] \right). \quad (3.3)$$

The last expression is closely related to the *relative rate of return* of wealth processes in \mathcal{X}_{++} , as the proof of the following result reveals.

Lemma 3.4 *Suppose that Y is a strictly positive predictable random field satisfying (Y1)–(Y3). Further, suppose that \tilde{p} is an I -valued, predictable, S -integrable process such that $\text{rel}^Y(p \mid \tilde{p}) = 0$ holds for all other I -valued predictable processes p . Define $\tilde{X} := \mathcal{E}(\int_0^{\cdot} \tilde{p}(t) dS(t))$. Then $\tilde{X}_0 = 1$, $\tilde{X} \in \mathcal{X}_{++}$, and X/\tilde{X} is a local \mathbb{P}^Y -martingale for all $X \in \mathcal{X}$.*

Proof Since \tilde{p} is S -integrable, \tilde{X} is well defined. In view of (3.3), the fact that $\text{rel}(0 \mid \tilde{p}) = 0$ implies that $\kappa^Y[\{x \in \mathbb{R} \mid \tilde{p}x = -1\}] = 0$. Therefore, $\tilde{p} \Delta S > -1$, i.e., $\tilde{X} > 0$ and $\tilde{X}_- > 0$ hold. With $\vartheta := \tilde{p} \tilde{X}_-$, we have $\tilde{X} = X^{1, \vartheta}$ in the notation of Sect. 2.1. Therefore, $\tilde{X} \in \mathcal{X}_{++}$.

Pick any $X = X^{x, \vartheta} \in \mathcal{X}_{++}$. Let $p := \vartheta/X_-$; then $X = x \mathcal{E}(\int_0^{\cdot} p(t) dS(t))$. We shall show that

$$\frac{X}{\tilde{X}} = x \frac{\mathcal{E}(\int_0^{\cdot} p(t) dS(t))}{\mathcal{E}(\int_0^{\cdot} \tilde{p}(t) dS(t))}$$

is a local \mathbb{P}^Y -martingale. Since $X > 0$, $X_- > 0$, $\tilde{X} > 0$ and $\tilde{X}_- > 0$ hold, it follows that we can write $X/\tilde{X} = x \mathcal{E}(R^{p \mid \tilde{p}})$ for some semimartingale $R^{p \mid \tilde{p}}$ with $\Delta R^{p \mid \tilde{p}} > -1$. In fact,

$$\begin{aligned} R^{p \mid \tilde{p}} &= \int_0^{\cdot} (p(t) - \tilde{p}(t)) dS(t) - \int_0^{\cdot} (p(t) - \tilde{p}(t)) \tilde{p}(t) d[S^c, S^c](t) \\ &\quad - \sum_{t \leq \cdot} \frac{(p(t) - \tilde{p}(t)) \tilde{p}(t) |\Delta S(t)|^2}{1 + \tilde{p}(t) \Delta S(t)}; \end{aligned}$$

indeed, using Yor's formula $\mathcal{E}(U)\mathcal{E}(V) = \mathcal{E}(U + V + [U, V])$, one easily checks that

$$\mathcal{E}\left(\int_0^{\cdot} \tilde{p}(t) dS(t)\right) \mathcal{E}(R^{p \mid \tilde{p}}) = \dots = \mathcal{E}\left(\int_0^{\cdot} p(t) dS(t)\right).$$

By a comparison of (3.3) with the formula for $R^{p \mid \tilde{p}}$ above, $\text{rel}^Y(p \mid \tilde{p}) = 0$ implies that $R^{p \mid \tilde{p}}$ is a \mathbb{P}^Y -sigma-martingale. (For information and properties of sigma-martingales, the reader is referred to [15].) Since $X/\tilde{X} = x \mathcal{E}(R^{p \mid \tilde{p}})$, it follows that X/\tilde{X} is a \mathbb{P}^Y -sigma-martingale. For nonnegative processes, the sigma-martingale property is equivalent to the local martingale property; therefore, we conclude that X/\tilde{X} is a local \mathbb{P}^Y -martingale.

Now, let $X \in \mathcal{X}$. Since $(1 + X)$ and 1 are in \mathcal{X}_{++} , the previous paragraph implies that $(1 + X)/\tilde{X}$ and $1/\tilde{X}$ are both local \mathbb{P}^Y -martingales. It follows that X/\tilde{X} is a local \mathbb{P}^Y -martingale. \square

In view of Lemma 3.4, Theorem 3.1 will be proved if we can find a strictly positive predictable random field Y satisfying (Y1)–(Y3), as well as an I -valued, predictable, S -integrable process \tilde{p}^Y such that $\text{rel}^Y(p \mid \tilde{p}^Y) = 0$ holds for any other I -valued predictable process p . In Sect. 3.2.4, we shall see how \tilde{p}^Y should be picked, given a strictly positive predictable random field Y satisfying (Y1)–(Y3); then, in Sect. 3.2.5, we shall construct the appropriate strictly positive predictable random field.

3.2.4 Growth rates

In order to understand how Y has to be picked, we shall use the fact that the relative rate of return is essentially the directional derivative of the growth rate. In more detail, define a predictable random field g^Y via $g^Y(p) = -\infty$ when $p \notin I$ and

$$g^Y(p) := pa^Y - (1/2)c^Y|p|^2 - \int_{\mathbb{R}} (px - \log(1 + px))\kappa^Y[\mathrm{d}x]$$

for $p \in I$. The assumption $\int_{\mathbb{R}} (|x| \wedge |x|^2)\kappa^Y[\mathrm{d}x] < \infty$ ensures that g is well defined and finite in the interior of I , though it might happen that $g^Y(\ell) = -\infty$ or $g^Y(r) = -\infty$. It is obvious that $g^Y(\omega, t, \cdot) : \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty\}$ is a concave function for fixed $(\omega, t) \in \Omega \times \mathbb{R}_+$. With all set inclusions involving subsets of $\Omega \times \mathbb{R}_+$ from now on to be understood in a $(\mathbb{P} \otimes G)$ -a.e. sense, an application of Lemma 3.2 (with a^Y and κ^Y replacing a and κ there, respectively) gives

$$\{r = \infty\} = \{\kappa^Y[(-\infty, 0)] = 0\} \subseteq \left\{ \lim_{p \rightarrow \infty} g^Y(p) \leq 0 \right\}.$$

Indeed, $\{\kappa^Y[(-\infty, 0)] = 0, c > 0\} \subseteq \{\lim_{p \rightarrow \infty} g^Y(p) = -\infty\}$, while

$$\{\kappa^Y[(-\infty, 0)] = 0, c = 0\} \subseteq \left\{ \lim_{p \rightarrow \infty} g^Y(p) = a - \int_{(0, \infty)} x\kappa[\mathrm{d}x] \right\}.$$

Similarly, one can show that $\{\ell = -\infty\} \subseteq \{\lim_{p \rightarrow -\infty} g^Y(p) \geq 0\}$. Since $g^Y(0) = 0$, it follows that g^Y always achieves its supremum at some point in I .

Define now the predictable random field $\nabla g^Y : \Omega \times \mathbb{R}_+ \times \mathbb{R} \rightarrow \mathbb{R} \cup \{-\infty, \infty\}$ (the formal derivative of g^Y) via

$$\begin{aligned} \nabla g^Y(p) &:= a^Y - pc^Y - \int_{\mathbb{R}} \frac{p|x|^2}{1 + px} \kappa^Y[\mathrm{d}x] \\ &= \nabla g(p) + \int_{\mathbb{R}} \frac{x}{1 + px} (Y(x) - 1)\kappa[\mathrm{d}x] \end{aligned} \quad (3.4)$$

for $p \in I$ (where $\nabla g := \nabla g^1$), $\nabla g^Y(p) = \nabla g^Y(\ell)$ for $p < \ell$ and similarly $\nabla g^Y(p) = \nabla g^Y(r)$ for $p > r$. The concavity of g^Y and straightforward applications of the dominated convergence theorem imply that for fixed $(\omega, t) \in \Omega \times \mathbb{R}_+$, ∇g is nonincreasing and continuous on I . Note that on $\{\ell = 0 = r\} = \{\text{supp}(\kappa) = \mathbb{R}\}$, it is impossible to define ∇g . In this case, we simply force $\nabla g^Y(p) = 0$ for all $p \in \mathbb{R}$; we shall see later how this convention is useful.

Define a process $\tilde{p}^Y := \inf\{p \in I \mid \nabla g^Y(p) \leq 0\}$, where we set $\tilde{p}^Y = r$ in case the last set is empty and $\tilde{p}^Y = 0$ on $\{\nabla g^Y(\ell) = 0 = \nabla g^Y(r)\}$. It is clear that \tilde{p}^Y is a predictable process. Furthermore, on $\{\nabla g^Y(\ell) \geq 0, \nabla g^Y(r) \leq 0\}$, which is a predictable set, we have $\nabla g^Y(\tilde{p}^Y) = 0$, and thus $\text{rel}^Y(p \mid \tilde{p}^Y) = (p - \tilde{p}^Y)\nabla g^Y(\tilde{p}^Y) = 0$ for all I -valued predictable processes p .

The point of the above discussion is the following. Suppose that for some strictly positive predictable random field Y satisfying (Y1)–(Y3), both $\nabla g^Y(\ell) \geq 0$ and $\nabla g^Y(r) \leq 0$ hold for all $(\omega, t) \in \Omega \times \mathbb{R}_+$, which as usual will be suppressed from the notation in the sequel. Then we can construct a predictable I -valued process \tilde{p}^Y such that $\text{rel}^Y(p \mid \tilde{p}^Y) = (p - \tilde{p}^Y)\nabla g^Y(\tilde{p}^Y) = 0$ for all I -valued predictable processes p . (Observe how $\text{rel}^Y(p \mid \tilde{p}^Y) = (p - \tilde{p}^Y)\nabla g^Y(\tilde{p}^Y) = 0$ trivially also holds on $\{\ell = 0 = r\} = \{\text{supp}(\kappa) = \mathbb{R}\}$ in view of our convention, as $I = \{0\}$.) In view of Lemma 3.4, Theorem 3.1 will follow as soon as we know that \tilde{p}^Y is S -integrable. Luckily, this is *always* the case under condition NA₁. The proof of this fact is quite technical, and basically follows the treatment in [16, Sect. 8], where Proposition 4.16 of the latter paper is proved. We shall, however, provide some details for completeness. In view of [1, Corollary 3.6.10], failure of S -integrability of \tilde{p}^Y implies that there exists a sequence of $[0, 1]$ -valued predictable processes $(h_n)_{n \in \mathbb{N}}$ such that each $h_n \tilde{p}^Y$, $n \in \mathbb{N}$, is S -integrable and the sequence of terminal values $(\int_0^T (h_n(t) \tilde{p}^Y(t)) dS(t))_{n \in \mathbb{N}}$ fails to be bounded in probability. (Note that a priori, the previous sequence can fail to be bounded in probability either from above or below, or even from both sides.) For each $n \in \mathbb{N}$, define $X_n \in \mathcal{X}_{++}$ with $X_n(0) = 1$ via

$$X_n := \mathcal{E}\left(\int_0^{\cdot} (h_n(t) \tilde{p}^Y(t)) dS(t)\right).$$

Since h_n is $[0, 1]$ -valued, the definition of \tilde{p}^Y implies that $\text{rel}^Y(0 \mid h_n \tilde{p}^Y) \leq 0$. (This follows because the predictable function $[0, 1] \ni u \mapsto g(u \tilde{p}^Y)$ is nondecreasing.) Therefore, $1/X_n$ is a nonnegative \mathbb{P} -supermartingale for all $n \in \mathbb{N}$. Then, it follows from [16, Lemma 8.1] that failure of boundedness in probability of the sequence $(\int_0^T (h_n(t) \tilde{p}^Y(t)) dS(t))_{n \in \mathbb{N}}$ also implies failure of boundedness in probability of the sequence $(X_n(T))_{n \in \mathbb{N}}$. (Although intuitively plausible, passing from failure of boundedness in probability of processes to failure of boundedness in probability of their stochastic exponentials is not always possible, because the stochastic exponential is not a monotone operator. The fact that this can be done in the present case is due to the fact that each process $1/X_n$ is a nonnegative \mathbb{P} -supermartingale—see also [16, Remark 8.2].) However, condition NA₁ is equivalent to the requirement that the set $\{X(T) \mid X \in \mathcal{X}$ with $X(0) = 1\}$ is bounded in probability, making it impossible for $(X_n(T))_{n \in \mathbb{N}}$ to fail to be bounded in probability. We conclude that \tilde{p}^Y is S -integrable under condition NA₁.

3.2.5 Construction of the appropriate predictable random field

We now move to the most technical part of the proof of Theorem 3.1, by constructing a strictly positive predictable random field Y satisfying (Y1)–(Y3), as well as the condition

$$(Y4) \quad \nabla g^Y(\ell) \geq 0 \text{ and } \nabla g^Y(r) \leq 0.$$

(Note that (Y4) is always trivially satisfied on $\{\ell = 0 = r\} = \{\text{supp}(\kappa) = \mathbb{R}\}$.) From the discussion of Sects. 3.2.3 and 3.2.4, existence of such a strictly positive predictable random field Y will complete the proof of Theorem 3.1.

The strictly positive predictable random field Y will actually depend on the predictable processes (a, κ, η) and will have to be defined differently on each of nine predictable sets $(P_i)_{i=1,\dots,9}$ that constitute a partition of $\Omega \times \mathbb{R}_+$. (By construction, it will be immediately clear that Y is actually a predictable random field.) On each of these predictable sets, we shall show that (Y1)–(Y4) are valid. The reader will notice how the one-dimensional structure of the asset price process is used in a non-trivial way when defining Y . The method certainly does not generalize for the case of multiple assets—it appears a big challenge to provide a proof in a multidimensional setting.

Before we delve into the technicalities of the proof, recall that under condition NA₁, any strictly positive predictable random field Y satisfying (Y1)–(Y3) is such that $\{\ell = -\infty\} \subseteq \{\nabla g^Y(\ell) \geq 0\}$ and $\{r = \infty\} \subseteq \{\nabla g^Y(r) \leq 0\}$. This is true in view of Lemma 3.2—see also the discussion in Sect. 3.2.4.

- We start with the set $P_1 := \{\ell = 0, r = \infty\}$. (All the predictable-set inclusions below are understood to hold on P_1 , until we move to the next case where they will be understood to hold on P_2 , and so forth.) Here, $\nabla g(\ell) = \nabla g(0) = a$. Since, as explained above, $\{r = \infty \subseteq \{\nabla g^Y(r) \leq 0\}\}$, we only have to carefully define Y on $\{a < 0\}$. Notice that $\{\ell = 0, r = \infty\} = \{\text{conv.supp}(\kappa) = [0, \infty)\}$, and define $Y_1 := y_1(a, \kappa, \eta)$, where, with

$$\delta := 1 + \frac{4}{\kappa[\mathbb{R}]} + \inf \left\{ x \in \mathbb{R} \mid \kappa[(0, x)] \geq \frac{\kappa[\mathbb{R}]}{2} \right\} \quad \text{and} \quad b := \left| s\delta - a + \frac{2}{\eta} \right|^2,$$

we set, for $x \in \mathbb{R}$,

$$y_1(a, \kappa, \eta; x) := 1 + \left(\frac{1}{\sqrt{b} \kappa[(b, \infty)]} \mathbb{I}_{(b, \infty)}(x) - \frac{1}{\sqrt{b} \kappa[(0, \delta)]} \mathbb{I}_{(0, \delta)}(x) \right) \mathbb{I}_{\{a < 0\}}.$$

(In the definition of $y_1(a, \kappa, \eta)$, the term $1/(\sqrt{b} \kappa[(0, \delta)])$ is understood to be zero on $\{\kappa[\mathbb{R}] = \infty\}$.) We shall show below that Y_1 satisfies (Y1)–(Y4). On $\{a \geq 0\}$ this is trivial, since $Y_1 = 1$. Therefore, we focus on $\{a < 0\}$ below. First of all, note that $Y_1 \geq 1/2$. Indeed, on $\{\kappa[\mathbb{R}] = \infty\}$ we have $Y_1 \geq 1$; also, on $\{\kappa[\mathbb{R}] < \infty\}$,

$$\sqrt{b} \kappa[(0, \delta)] > \delta \kappa[(0, \delta)] > \frac{4}{\kappa[\mathbb{R}]} \frac{\kappa[\mathbb{R}]}{2} = 2$$

holds from the definition of δ . Proceeding, the fact that Y_1 is bounded from above coupled with $\int_{\mathbb{R}} (|x| \wedge |x|^2) \kappa[\text{d}x] < \infty$ implies $\int_{\mathbb{R}} (|x| \wedge |x|^2) Y_1(x) \kappa[\text{d}x] < \infty$. For the estimate of the distance between κ and κ^{Y_1} , observe that

$$\int_{\mathbb{R}} |Y_1(x) - 1| \kappa[\text{d}x] \leq \frac{2}{\sqrt{b}} \leq \frac{2}{2/\eta} = \eta.$$

Now, on $\{\kappa[\mathbb{R}] = \infty\}$, we have $Y_1 \geq 1$ and obviously $\kappa^{Y_1}[\mathbb{R}] = \infty$; on the other hand, on $\{\kappa[\mathbb{R}] < \infty\}$, the equality $\kappa^{Y_1}[\mathbb{R}] = \kappa[\mathbb{R}]$ follows in a straightforward

way from the definition of Y_1 . Finally, since $\nabla g(0) = a$, use (3.4) to estimate

$$\begin{aligned}\nabla g^{Y_1}(0) &= a + \int_{(b, \infty)} \frac{x}{\sqrt{b}\kappa[(b, \infty)]} \kappa[\mathrm{d}x] - \int_{(0, \delta)} \frac{x}{\sqrt{b}\kappa[(0, \delta)]} \kappa[\mathrm{d}x] \\ &\geq a + \sqrt{b} - \frac{\delta}{\sqrt{b}} = a - a + 2/\eta + \delta - \frac{\delta}{\delta - a + 2/\eta} \geq 0.\end{aligned}$$

(The last inequality follows from $\eta > 0$ and $\delta > 1$, which also imply that $\delta - a + 2/\eta > 1$, since $a < 0$.)

- The situation on $P_2 := \{\ell = -\infty, r = 0\}$ is symmetric to the previous one. With

$$\delta := 1 + \frac{4}{\kappa[\mathbb{R}]} - \sup \left\{ x \in \mathbb{R} \mid \kappa[(x, 0)] \geq \frac{\kappa[\mathbb{R}]}{2} \right\} \quad \text{and} \quad b := \left| \delta + a + \frac{2}{\eta} \right|^2,$$

define $Y_2 := y_2(a, \kappa, \eta)$, where for $x \in \mathbb{R}$,

$$\begin{aligned}y_2(a, \kappa, \eta; x) &:= 1 + \left(\frac{1}{\sqrt{b}\kappa[(-\infty, -b)]} \mathbb{I}_{(-\infty, -b)}(x) \right. \\ &\quad \left. - \frac{1}{\sqrt{b}\kappa[[-\delta, 0]]} \mathbb{I}_{[-\delta, 0]}(x) \right) \mathbb{I}_{\{a > 0\}}.\end{aligned}$$

One can then follow the exact same steps that we carried out on P_1 .

- We now move to the set $P_3 := \{\ell = -\infty, 0 < r < \infty\}$, on which we have $\text{conv.supp}(\kappa) = [-1/r, 0]$. Since $\ell = -\infty$, $\nabla g(\ell) \geq 0$. Also, on $\{\kappa[\{-1/r\}] > 0\}$, we have $g(r) = -\infty$, and $\nabla g(r) = -\infty$ follows easily. Define $Y_3 := y_3(a, \kappa, \eta)$, where, with

$$\beta := \frac{1}{r} \min \left\{ \frac{1}{2}, \exp \left(-\frac{2r}{\kappa[\mathbb{R}]} \right), \exp \left(-\frac{2r}{\eta} \right) \right\},$$

$y_3(a, \kappa, \eta; x)$ is for all $x \in \mathbb{R}$ equal to

$$\begin{aligned}1 + &\left(\frac{r}{\kappa[\mathbb{R}] \log(r\beta)} + \mathbb{I}_{(-\frac{1}{r}, \beta - \frac{1}{r})}(x) \right. \\ &\times \left. \int_x^{\beta - \frac{1}{r}} \frac{|r|^2}{(1 + rw)|\log(1 + rw)|^2 \kappa[(-\frac{1}{r}, w)]} \mathrm{d}w \right) \mathbb{I}_{\{\kappa[\{-\frac{1}{r}\}] = 0\}}.\end{aligned}$$

Since $\log(r\beta) \leq -2r/\kappa[\mathbb{R}]$, we easily get $Y_3 \geq 1/2 > 0$. On $\{\kappa[\mathbb{R}] = \infty\}$, we have $Y_3 \geq 1$ and $\kappa^{Y_3}[\mathbb{R}] = \infty$ trivially follows; on the other hand, on $\{\kappa[\mathbb{R}] < \infty\}$, $\kappa^{Y_3}[\mathbb{R}] = \kappa[\mathbb{R}]$ follows as soon as one notices that the double integral

$$\int_{(-1/r, \beta - 1/r]} \left(\int_x^{\beta - 1/r} \frac{|r|^2}{(1 + rw)|\log(1 + rw)|^2 \kappa[(-1/r, w)]} \mathrm{d}w \right) \kappa[\mathrm{d}x]$$

is by Fubini's theorem equal to

$$\int_{-1/r}^{\beta - 1/r} \frac{|r|^2}{(1 + rw)|\log(1 + rw)|^2} \mathrm{d}w = r \int_0^{r\beta} \frac{1}{w|\log w|^2} \mathrm{d}w = -\frac{r}{\log(r\beta)}. \quad (3.5)$$

The above estimate also implies $\int_{\mathbb{R}} (|x| \wedge |x|^2) Y_3(x) \kappa(dx) < \infty$. Indeed, note that

$$Y_3(x) \leq 1 + r / (\kappa[\mathbb{R}] \log(r\beta))$$

for $x \in I \setminus (-1/r, \beta - 1/r]$, while $\beta \leq 1/(2r)$ implies that

$$\int_{(-1/r, \beta - 1/r]} (|x| \wedge |x|^2) Y_3(x) \kappa(dx) \leq \frac{1}{r \min\{1, r\}} \int_{(-1/r, \beta - 1/r]} Y_3(x) \kappa(dx)$$

is finite. For estimating the distance between κ and κ^{Y_3} , note that

$$\int_{\mathbb{R}} |Y_3(x) - 1| \kappa(dx) \leq -2r / \log(r\beta) \leq \eta,$$

which follows from the definition of β and the calculations that lead to (3.5). We shall now show that $g^{Y_3}(r) = -\infty$, therefore establishing that $\nabla g^{Y_3}(r) \leq 0$. Start with the observation that, for $x \in (-1/r, \beta - 1/r]$, integration by parts gives

$$\begin{aligned} \log(1 + rx) Y_3(x) &= \log(r\beta) + \frac{r}{\kappa[\mathbb{R}]} - \int_x^{\beta-1/r} \frac{r}{1 + rw} Y_3(w) dw \\ &\quad + \int_x^{\beta-1/r} \frac{|r|^2}{(1 + rw) \log(1 + rw) \kappa[(-1/r, w)]} dw \\ &\leq \frac{r}{\kappa[\mathbb{R}]} + \int_x^{\beta-1/r} \frac{|r|^2}{(1 + rw) \log(1 + rw) \kappa[(-1/r, w)]} dw. \end{aligned}$$

By the above estimate and Fubini's theorem, $\int_{(-1/r, \beta - 1/r]} \log(1 + rx) Y_3(x) \kappa(dx)$ is bounded from above by the quantity

$$\frac{r\kappa[(-1/r, \beta - 1/r)]}{\kappa[\mathbb{R}]} + |r|^2 \int_{-1/r}^{\beta-1/r} (1 + rw)^{-1} \log^{-1}(1 + rw) dw = -\infty.$$

This last fact together with (3.4) and $\int_{\mathbb{R}} (|x| \wedge |x|^2) \kappa(dx) < \infty$ gives $g^{Y_3}(r) = -\infty$. Of course, $\nabla g^{Y_3}(\ell) \geq 0$ follows because $\ell = -\infty$.

- The situation on $P_4 := \{-\infty < \ell < 0, r = \infty\}$ is symmetric to P_3 , and therefore details will be omitted. Just define $Y_4 := y_4(a, \kappa, \eta)$, where, with

$$\beta := \frac{1}{\ell} \min \left\{ \frac{1}{2}, \exp \left(\frac{2\ell}{\kappa[\mathbb{R}]} \right), \exp \left(\frac{2\ell}{\eta} \right) \right\},$$

$y_4(a, \kappa, \eta; x)$ is for all $x \in \mathbb{R}$ equal to

$$1 + \left(\frac{\ell}{\kappa[\mathbb{R}] \log(\ell\beta)} + \mathbb{I}_{(\beta-\frac{1}{\ell}, -\frac{1}{\ell})}(x) \right. \\ \left. \times \int_{\beta-\frac{1}{\ell}}^x \frac{|\ell|^2}{(1+\ell w) |\log(1+\ell w)|^2 \kappa[[w, -\frac{1}{\ell}]]} dw \right) \mathbb{I}_{\{\kappa[-\frac{1}{\ell}] = 0\}}.$$

- We now move to $P_5 := \{\ell = 0, 0 < r < \infty\}$. Here, we shall use a combination of the work we carried out for P_1 and P_3 . Remembering the definitions of the deterministic functionals y_1 and y_3 , define

$$Y_5 := y_1(a^{y_3(a, \kappa, \eta/2)}, \kappa^{y_3(a, \kappa, \eta/2)}, \eta/2) y_3(a, \kappa, \eta/2).$$

The definition of Y_5 is essentially realized in two steps. First there is a change according to y_3 . This forces $g^{y_3(a, \kappa, \eta/2)}(r) = -\infty$ as on P_3 . Also, (Y1), (Y2) and (Y3) hold, with $\eta/2$ replacing η in (Y2). In the second step there is a change using y_1 . Since $y_1(a^{y_3(a, \kappa, \eta/2)}, \kappa^{y_3(a, \kappa, \eta/2)}, \eta/2; x) = 1$ for all $x \in (-\infty, 0)$, $g^{Y_5}(r) = -\infty$ (and, therefore, $\nabla g^{Y_5}(r) \leq 0$) still holds, while now it is also the case that $\nabla g^{Y_5}(\ell) \geq 0$, as was the case on P_1 . It is clear that $Y_5 > 0$ (since both of the predictable random fields appearing in the definition of Y_5 are strictly positive), and that (Y1)–(Y4) all hold.

- On $P_6 := \{-\infty < \ell < 0, r = 0\}$, define

$$Y_6 := y_2(a^{y_4(a, \kappa, \eta/2)}, \kappa^{y_4(a, \kappa, \eta/2)}, \eta/2) y_4(a, \kappa, \eta/2).$$

The situation is symmetric to the one on P_5 —just follow the exact same reasoning.

- Moving to $P_7 := \{-\infty < \ell < 0 < r < \infty\}$, we shall use a combination of the treatment on P_3 and P_4 . Define

$$Y_7 := y_3(a^{y_4(a, \kappa, \eta/2)}, \kappa^{y_4(a, \kappa, \eta/2)}, \eta/2) y_4(a, \kappa, \eta/2).$$

The validity of (Y1)–(Y4) follows by the same reasoning as on the set P_5 .

- On $P_8 := \{\ell = 0, r = 0\} \subseteq \{\nabla g(0) = 0\}$, there is no need to do anything; simply set $Y_8 := 1$.
- Finally, on $P_9 := \{\ell = -\infty, r = \infty\} = \{\text{conv.supp}(\kappa) = \emptyset\}$, there is also no need to do anything; set $Y_9 := 1$. Indeed, we either have $c = 0$, which implies that $a = 0$ and therefore $\nabla g(-\infty) = \nabla g(+\infty) = 0$, or $c > 0$, in which case $\nabla g(-\infty) = \infty$ and $\nabla g(+\infty) = -\infty$.

Acknowledgements The author would like to thank Yuri Kabanov for fruitful conversations that significantly helped in formulating and proving the results of this paper. Two anonymous referees provided invaluable help in the presentation of the paper.

References

1. Bichteler, K.: Stochastic Integration with Jumps. Encyclopedia of Mathematics and Its Applications, vol. 89. Cambridge University Press, Cambridge (2002)
2. Brannath, W., Schachermayer, W.: A bipolar theorem for $L_+^0(\Omega, \mathcal{F}, \mathbf{P})$. In: Séminaire de Probabilités, XXXIII. Lecture Notes in Math., vol. 1709, pp. 349–354. Springer, Berlin (1999)

3. Dalang, R.C., Morton, A., Willinger, W.: Equivalent martingale measures and no-arbitrage in stochastic securities market models. *Stoch. Stoch. Rep.* **29**, 185–201 (1990)
4. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. *Math. Ann.* **300**, 463–520 (1994)
5. Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. *Math. Ann.* **312**, 215–250 (1998)
6. Fernholz, E., Karatzas, I.: Stochastic portfolio theory: an overview. In: Bensoussan, A. (ed.) *Handbook of Numerical Analysis. Mathematical Modeling and Numerical Methods in Finances*, pp. 89–168 (2009)
7. Föllmer, H., Kabanov, Y.M.: Optional decomposition and Lagrange multipliers. *Finance Stoch.* **2**, 69–81 (1998)
8. Gilles, C., LeRoy, S.F.: Bubbles and charges. *Int. Econ. Rev.* **33**, 323–339 (1992)
9. Harrison, J.M., Kreps, D.M.: Martingales and arbitrage in multiperiod securities markets. *J. Econ. Theory* **20**, 381–408 (1979)
10. Ingersoll, J.E.: *Theory of Financial Decision Making*. Rowman & Littlefield Studies in Financial Mathematics. Rowman & Littlefield, Totowa (1987)
11. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 288. Springer, Berlin (2003)
12. Kabanov, Y.M.: On the FTAP of Kreps–Delbaen–Schachermayer. In: Kabanov, Y.M., Rozovskii, B.L., Shiryaev, A.N. (eds.) *Statistics and Control of Stochastic Processes*, Moscow, 1995/1996, pp. 191–203. World Sci. Publ., River Edge (1997)
13. Kabanov, Y.M., Kramkov, D.O.: Large financial markets: asymptotic arbitrage and contiguity. *Teor. Veroâtn. ee Primen.* **39**, 222–229 (1994)
14. Kabanov, Y.M., Liptser, R.S., Shiryaev, A.N.: Absolute continuity and singularity of locally absolutely continuous probability distributions. (I). *Mat. Sb.* **107**, 364–415 (1978)
15. Kallsen, J.: σ -localization and σ -martingales. *Teor. Veroâtn. ee Primen.* **48**, 177–188 (2003)
16. Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. *Finance Stoch.* **11**, 447–493 (2007)
17. Kardaras, C.: Finitely additive probabilities and the fundamental theorem of asset pricing. In: Chiarella, C., Novikov, A. (eds.) *Contemporary Quantitative Finance: Essays in Honour of Eckhard Platen*, pp. 19–34. Springer, Berlin (2010)
18. Loewenstein, M., Willard, G.A.: Local martingales, arbitrage, and viability. Free snacks and cheap thrills. *Econom. Theory* **16**, 135–161 (2000)
19. Loewenstein, M., Willard, G.A.: Rational equilibrium asset-pricing bubbles in continuous trading models. *J. Econ. Theory* **91**, 17–58 (2000)
20. Platen, E., Heath, D.: *A Benchmark Approach to Quantitative Finance*. Springer Finance. Springer, Berlin, (2006)
21. Rogers, L.C.G.: Equivalent martingale measures and no-arbitrage. *Stoch. Stoch. Rep.* **51**, 41–49 (1994)
22. Rokhlin, D.B.: Asymptotic arbitrage and numéraire portfolios in large financial markets. *Finance Stoch.* **12**, 173–194 (2008)
23. Schweizer, M.: On the minimal martingale measure and the Föllmer–Schweizer decomposition. *Stoch. Anal. Appl.* **13**, 573–599 (1995)