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No arbitrage of the first kind and local martingale numéraires

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Abstract A supermartingale deflator (resp. local martingale deflator) multiplicatively transforms nonnegative wealth processes into supermartingales (resp. local martingales). A supermartingale numéraire (resp. local martingale numéraire) is a wealth process whose reciprocal is a supermartingale deflator (resp. local martingale deflator). It has been established in previous works that absence of arbitrage of the first kind (NA_1) is equivalent to the existence of the (unique) supermartingale numéraire, and further equivalent to the existence of a strictly positive local martingale deflator; however, under NA_1, a local martingale numéraire may fail to exist. In this work, we establish that under NA_1, a supermartingale numéraire under the original probability P becomes a local martingale numéraire for equivalent probabilities arbitrarily close to P in the total variation distance.

Keywords Arbitrage · Viability · Fundamental theorem of asset pricing · Numéraire · Local martingale deflator · σ-martingale

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1 Introduction

A central structural assumption in the mathematical theory of financial markets is the existence of so-called *local martingale deflators*, that is, processes that act multiplicatively and transform nonnegative wealth processes into local martingales. Under the *no free lunch with vanishing risk* (NFLVR) condition of [5, 6], the density process of a local martingale (or, more generally, a σ-martingale) measure is a strictly positive local martingale deflator. However, strictly positive local martingale deflators may exist even if the market allows a free lunch with vanishing risk. Both from a financial and mathematical points of view, an especially important case occurs when a deflator is the reciprocal of a wealth process called a *local martingale numéraire*; in this case, the prices of all assets (and in fact, all wealth processes resulting from trading), when denominated in units of the latter local martingale numéraire, are (positive) local martingales.

The relevant, weaker than NFLVR, market viability property, which turns out to be equivalent to the existence of supermartingale (or local martingale) numéraires, was isolated by various authors under different names: *no asymptotic arbitrage of the first kind* (NAA_1), *no arbitrage of the first kind* (NA_1), *no unbounded profit with bounded risk* (NUPBR), and so on; see [10, 5, 9, 12, 14]. It is not difficult to show that all these properties are in fact equivalent, even in a wider framework than that of the standard semimartingale setting—for more information, see the Appendix. In the present paper, we opt to utilize the economically meaningful formulation NA_1, defined as the property of the market to assign a strictly positive superhedging value to any non-trivial positive contingent claim.

In the standard financial model studied here, the market is described by a d-dimensional semimartingale process S giving the discounted prices of basic securities. In [12], it was shown (even in the more general case of convex portfolio constraints) that the following statements are equivalent:

(i) *Condition* NA_1 *holds.*

(ii) *There exists a strictly positive supermartingale deflator.*

(ii') *The (unique) supermartingale numéraire exists.*

In [17], the previous list of equivalent properties was complemented by

(iii) *There exists a strictly positive local martingale deflator.*

There are counterexamples (see e.g. [17]) showing that the local martingale numéraire may fail to exist even when there is an equivalent martingale measure (and in particular when condition NA_1 holds). Such examples are possible only in the case of discontinuous asset price processes; it was already shown in [2] that for continuous semimartingales, among all strictly positive local martingale deflators, there exists one whose reciprocal is a strictly positive wealth process.

In the present note, we add to the above list of equivalences a further property:

(iv) *In any total-variation neighbourhood of the original probability, there exists an equivalent probability under which the (unique) local martingale numéraire exists.*

Establishing the chain (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) $\Rightarrow$ (i) is rather straightforward and well known. The contribution of the note is proving the "closing" implication (i) $\Rightarrow$ (iv). It is an obvious corollary of the already known implication (i) $\Rightarrow$ (ii') and the following principal result of our note, a version of which was established previously only for the case $d = 1$ in [14].

Proposition 1.1 *A supermartingale numéraire under P becomes a local martingale numéraire under probabilities $\tilde{P} \sim P$ that are arbitrarily close in the total-variation distance to P.*

Proposition 1.1 bears a striking similarity with the density result of σ-martingale measures in the set of all separating measures—see [6] and Theorem A.5 in the Appendix. In fact, coupled with certain rather elementary properties of stochastic exponentials, the aforementioned density result is the main ingredient of our proof of Proposition 1.1.

Importantly, we recover in particular the main result of [17], utilising completely different arguments. The proof in [17] combines a change-of-numéraire technique and a reduction to the Delbaen–Schachermayer fundamental theorem of asset pricing (FTAP) in [5]. The latter is considered as one of the most difficult and fundamental results of arbitrage theory, and the search for simplified proofs still continues—see, for example, [3]. In fact, it may be obtained as a by-product of our result and the version of the optional decomposition theorem in [16], as has been explained in [13, Sect. 3].

2 Framework and main result

2.1 The setup

In all that follows, we fix $T \in (0, \infty)$ and work on a filtered probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\in[0,T]}, P)$ satisfying the usual conditions. Unless otherwise explicitly specified, all relationships between random variables are understood in the P-a.s. sense, and all relationships between stochastic processes are understood modulo P-evanescence.

Let $S = (S_t)_{t\in[0,T]}$ be a d-dimensional semimartingale. We denote by $L(S)$ the set of S-integrable processes, that is, the set of all d-dimensional predictable processes for which the stochastic integral $H \cdot S$ is defined. We stress that we consider general vector stochastic integration—see [8, Chapter III, Sect. 6].

An integrand $H \in L(S)$ such that $x + H \cdot S \geq 0$ for some $x \in \mathbb{R}_+$ will be called *x-admissible*. We introduce the set of semimartingales

$$\mathcal{X}^x := \{H \cdot S : \ H \text{ is } x\text{-admissible integrand}\},$$

and denote $\mathcal{X}^x_>$ its subset formed by the processes X such that $x + X > 0$ and $x + X_- > 0$. These sets are invariant under equivalent changes of the underlying probability. Define also the sets of random variables $\mathcal{X}^x_T := \{X_T : \ X \in \mathcal{X}^x\}$.

For $\xi \in L^0_+$, we define

$$x(\xi) := \inf\{x \in \mathbb{R}_+ : \text{ there exists } X \in \mathcal{X}^x \text{ with } x + X_T \geq \xi\},$$

with the standard convention $\inf \emptyset = \infty$.

In the special context of financial modelling:

- The process S represents the price evolution of d liquid assets, discounted by a certain baseline security labelled 0 or $d+1$.
- With H being an x-admissible integrand, $x + H \cdot S$ is the value process of a self-financing portfolio with the initial capital $x \geq 0$, constrained to stay nonnegative at all times.
- A random variable $\xi \in L^0_+$ represents a contingent claim, and $x(\xi)$ is its *super-hedging value* in the class of nonnegative wealth processes.

2.2 Main result

We define $|P - \tilde{P}|_{TV} = \sup_{A \in \mathcal{F}} |P[A] - \tilde{P}[A]|$ as the total-variation distance between the probabilities P and $\tilde{P}$ on $(\Omega, \mathcal{F})$.

Theorem 2.1 *The following conditions are equivalent*:

(i) $x(\xi) > 0$ *for every* $\xi \in L^0_+ \setminus \{0\}$.

(ii) *There exists a strictly positive process* Y *such that the process* $Y(1+X)$ *is a supermartingale for every* $X \in \mathcal{X}^1$.

(iii) *There exists a strictly positive process* Y *with* $Y_0 \in L^1$ *such that the process* $Y(1+X)$ *is a local martingale for every* $X \in \mathcal{X}^1$.

(iv) *For any* $\varepsilon > 0$, *there exists* $\tilde{P} \sim P$ *with* $|\tilde{P} - P|_{TV} < \varepsilon$ *and* $\tilde{X} \in \mathcal{X}^1_>$ *such that* $(1+X)/(1+\tilde{X})$ *is a local* $\tilde{P}$*-martingale for every* $X \in \mathcal{X}^1$.

Remark 2.2 It is straightforward to check that statements (ii), (iii), and (iv) of Theorem 2.1 are equivalent to the same conditions where "for every $X \in \mathcal{X}^1$" is replaced by "for every $X \in \mathcal{X}^1_>$".

Theorem 2.1 is formulated in the "pure" language of stochastic analysis. In the context of mathematical finance, the following interpretations regarding its statement should be kept in mind:

- Condition (i) states that any non-trivial contingent claim $\xi \geq 0$ has a strictly positive superhedging value. This is referred to as condition NA_1 (no arbitrage of the first kind); it is equivalent to the boundedness in probability of the set $\mathcal{X}^1_T$ or, alternatively, to condition NAA_1 (no asymptotic arbitrage of the first kind)—see the Appendix.
- The process Y in statement (ii) (resp. in statement (iii)) is called a strictly positive *supermartingale deflator* (resp. *local martingale deflator*).
- A process $\tilde{X}$ with the property in statement (iv) is called a *local martingale numéraire under the probability* $\tilde{P}$. It is well known and not hard to check that a local martingale numéraire (as well as a supermartingale numéraire) is unique if it exists.

With the above terminology in mind, we may reformulate the properties (i)–(iv) as was done in the Introduction.

3 Proof of Theorem 2.1

3.1 Proof of easy implications

The arguments establishing the implications (iv) ⇒ (iii) ⇒ (ii) ⇒ (i) in Theorem 2.1 are elementary and well known; however, for completeness of presentation, we briefly reproduce them here.

Assume statement (iv), and in its notation fix some $\varepsilon > 0$. Let Z be the density process of $\tilde{P}$ with respect to P, and set $\tilde{Z} := 1/(1+\tilde{X})$. For any $X \in \mathcal{X}^1$, the process $\tilde{Z}(1+X)$ is a local $\tilde{P}$-martingale. Hence, with $Y := Z\tilde{Z}$, the process $Y(1+X)$ is a local P-martingale, that is, (iii) holds.

Since a positive local martingale with initial value in L^1 is a supermartingale, the implication (iii) ⇒ (ii) is obvious.

To establish the implication (ii) ⇒ (i), suppose that Y is a strictly positive supermartingale deflator. It follows that $EY_T(1+X_T) \le 1$ for all $X \in \mathcal{X}^1$. Hence, the set $Y_T(1+\mathcal{X}^1_T)$ is bounded in L^1 and, a fortiori, bounded in probability. Since $Y_T > 0$, the set $\mathcal{X}^1_T$ is also bounded in probability. The latter property is equivalent to condition NA_1—see Lemma A.1 in the Appendix.

By [12, Theorem 4.12] and Lemma A.1 in the Appendix, condition (i) in the statement of Theorem 2.1 implies the existence of the (unique) supermartingale numéraire. Therefore, in order to establish the implication (i) ⇒ (iv) of Theorem 2.1 and complete its proof, it remains to prove Proposition 1.1. For this, we need some auxiliary facts presented in the next subsection.

3.2 Ratios of stochastic exponentials

We introduce the notation

$$B(S) := \{f \in L(S) \colon f\Delta S > -1\};$$

that is, $B(S)$ is the subset of integrands for which the trajectories of the stochastic exponentials $\mathcal{E}(f \cdot S)$ are bounded away from zero.

Note that the set $1+\mathcal{X}^1_>$ coincides with the set of stochastic exponentials of integrals with respect to S, that is,

$$1+\mathcal{X}^1_> = \{\mathcal{E}(f \cdot S) : f \in B(S)\}.$$

Indeed, the stochastic exponential corresponding to an integrand $f \in B(S)$ is strictly positive, and so is its left limit, and it satisfies the linear integral equation

$$\mathcal{E}(f \cdot S) = 1 + \mathcal{E}_-(f \cdot S) \cdot (f \cdot S) = 1 + \big(\mathcal{E}_-(f \cdot S) f\big) \cdot S.$$

Thus, $\mathcal{E}(f \cdot S) \in \mathcal{X}^1_>$. Conversely, if the process $V = 1 + H \cdot S$ is such that $V > 0$ and $V_- > 0$, then

$$V = 1 + (V_- V_-^{-1}) \cdot V = 1 + V_- \cdot \big(V_-^{-1} \cdot (H \cdot S)\big) = 1 + V_- \cdot \big((V_-^{-1} H) \cdot S\big);$$

that is, $V = \mathcal{E}(f \cdot S)$, where $f = V_-^{-1} H \in B(S)$ because $f\Delta S = V_-^{-1}\Delta V > -1$.

The above observations, coupled with Remark 2.2, imply that condition (iv) may be alternatively reformulated as follows:

(iv) *For any* $\varepsilon > 0$, *there exist* $g \in B(S)$ *and* $\tilde{P} \sim P$ *with* $|\tilde{P} - P|_{TV} < \varepsilon$ *such that* $\mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S)$ *is a local* $\tilde{P}$*-martingale for every* $f \in B(S)$.

Let S^c denote the continuous local martingale part of the semimartingale S. Recall that $\langle S^c \rangle = c \cdot A$, where A is a predictable increasing process, and c is a predictable process with values in the set of positive semidefinite matrices; then, $g \in L(S^c)$ if and only if $|c^{1/2} g|^2 \cdot A_T < \infty$.

In the sequel, fix an arbitrary $g \in B(S)$ and set

$$S^g = S - cg \cdot A - \sum_{s \le \cdot} \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \Delta S_s. \tag{3.1}$$

As

$$\sum_{s \le \cdot} \left| \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \right| |\Delta S_s| \le \frac{1}{2} \sum_{s \le T} \left| \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \right|^2 + \frac{1}{2} \sum_{s \le T} |\Delta S_s|^2 < \infty,$$

the last term on the right-hand side of (3.1) is a process of finite variation, implying that S^g is a semimartingale. (Recall that $g \cdot S$ is a semimartingale, so that on any finite interval, there are P-a.s. only finitely many s with $|g_s \Delta S_s| \ge 1/2$.)

Noting that $\Delta S^g = \Delta S/(1 + g \Delta S)$, we obtain from (3.1) that

$$S = S^g + cg \cdot A + \sum_{s \le \cdot} (g_s \Delta S_s) \Delta S^g_s.$$

Lemma 3.1 $L(S) = L(S^g)$.

Proof Let $f \in L(S)$. Then

$$|(f, cg)| \cdot A_T \le \frac{1}{2} |c^{1/2} f|^2 \cdot A_T + \frac{1}{2} |c^{1/2} g|^2 \cdot A_T < \infty$$

and

$$\sum_{s \le T} \frac{|g_s \Delta S_s f_s \Delta S_s|}{1 + g_s \Delta S_s} \le \frac{1}{2} \sum_{s \le T} |f_s \Delta S_s|^2 + \frac{1}{2} \sum_{s \le T} \left| \frac{g_s \Delta S_s}{1 + g_s \Delta S_s} \right|^2 < \infty.$$

Thus, $L(S) \subseteq L(S^g)$. To show the opposite inclusion, take $f \in L(S^g)$. The conditions $g \in L(S)$ and $f \in L(S^g)$ imply that f and g are integrable with respect to $S^c = (S^g)^c$, that is, $|c^{1/2} g|^2 \cdot A_T < \infty$ and $|c^{1/2} f|^2 \cdot A_T < \infty$. As previously, it then follows that $|(f, cg)| \cdot A_T < \infty$. Since also

$$\sum_{s \le t} |(g_s \Delta S_s)(f_s \Delta S^g_s)| \le \frac{1}{2} \sum_{s \le T} |g_s \Delta S_s|^2 + \frac{1}{2} \sum_{s \le T} |f_s \Delta S^g_s|^2 < \infty,$$

we obtain that $f \in L(S)$, that is, the inclusion $L(S^g) \subseteq L(S)$. □

Lemma 3.2 $B(S^g) = B(S) - g$.

Proof Let $h = f - g$, where $f \in B(S)$. Then, $h \in L(S) = L(S^g)$ by Lemma 3.1, and

$$h\Delta S^g = (f-g)\Delta S^g = \frac{(f-g)\Delta S}{1+g\Delta S} = \frac{1+f\Delta S}{1+g\Delta S} - 1 > -1.$$

Conversely, start with $h \in B(S^g)$. Then, using again Lemma 3.1, we obtain that $f := h+g$ belongs to $L(S)$. Moreover, recalling the relation $\Delta S = \Delta S^g/(1-g\Delta S^g)$, we obtain that

$$f\Delta S = (h+g)\Delta S = \frac{(h+g)\Delta S^g}{1-g\Delta S^g} = \frac{1+h\Delta S^g}{1-g\Delta S^g} - 1 > -1,$$

which completes the proof. □

For $f \in B(S)$, Lemma 3.2 gives $f - g \in B(S^g)$; then, straightforward calculations using Yor's product formula

$$\mathcal{E}(U)\mathcal{E}(V) = \mathcal{E}(U + V + [U, V]),$$

which is valid for arbitrary semimartingales U and V, lead to the identity

$$\frac{\mathcal{E}(f \cdot S)}{\mathcal{E}(g \cdot S)} = \mathcal{E}\big((f-g)\cdot S^g\big).$$

(In this regard, see also [12, Lemma 3.4].) Then, invoking Lemma 3.2, we obtain the set equality

$$1 + \mathcal{X}^1_>(S^g) = \mathcal{E}^{-1}(g \cdot S)\big(1 + \mathcal{X}^1_>(S)\big).$$

3.3 Proof of Proposition 1.1

Let the process $\mathcal{E}(g \cdot S)$, where $g \in B(S)$, be the (unique) supermartingale numéraire; in other words, the ratio $\mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S)$ is a supermartingale for each $f \in B(S)$. Passing to S^g and using Lemma 3.2, we obtain that $E\mathcal{E}_T(h \cdot S^g) \le 1$ for all $h \in B(S^g)$. Therefore, $EH \cdot S^g_T \le 0$ for every $H \in L(S^g)$ such that $H \cdot S^g > -1$ and $H \cdot S^g_- > -1$ and thus for every $H \in L(S^g)$ for which the process $H \cdot S^g$ is bounded from below. This means, in the terminology of [9], that the probability P is a separating measure for S^g. An application of [6, Proposition 4.7] (or Theorem A.5) implies, for any $\varepsilon > 0$, the existence of probability $\tilde{P} \sim P$, depending on ε, with $|P - \tilde{P}|_{TV} < \varepsilon$ and such that S^g is a σ-martingale with respect to $\tilde{P}$, that is, $S^g = G \cdot M$ where G is a $(0, 1]$-valued one-dimensional predictable process and M is a d-dimensional local $\tilde{P}$-martingale. Recall that a bounded from below stochastic integral with respect to a local martingale is a local martingale [1, Proposition 3.3]. The ratio $\mathcal{E}(f \cdot S)/\mathcal{E}(g \cdot S)$, being an integral with respect to S^g, hence with respect to M, is a $\tilde{P}$-local martingale for each $f \in B(S)$, which is exactly what we need. □

Remark 3.1 An inspection of the arguments in [12] used to establish the implication (i) ⇒ (ii') reveals that in the case where the (random, (w, t)-dependent) Lévy measures of S are concentrated on finite sets also depending on (ω, t), the (unique) supermartingale numéraire is in fact the (unique) local martingale numéraire.

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Appendix: No-arbitrage conditions, revisited

A.1 Condition NA_1: equivalent formulations

We discuss equivalent forms of the condition NA_1 in the context of a general abstract setting, where the model is given by specifying the set of wealth processes. The advantage of this generalization is that we can use only elementary properties without any reference to stochastic calculus and integration theory.

Let $\mathcal{X}^1$ be a convex set of càdlàg processes X with $X \geq -1$ and $X_0 = 0$, containing the zero process. For $x \geq 0$, we define the set $\mathcal{X}^x = x\mathcal{X}^1$ and note that $\mathcal{X}^x \subseteq \mathcal{X}^1$ when $x \in [0, 1]$. Put $\mathcal{X} := \operatorname{cone} \mathcal{X}^1 = \mathbb{R}_+ \mathcal{X}^1$ and define the sets of terminal random variables $\mathcal{X}^1_T := \{X_T : \; X \in \mathcal{X}^1\}$ and $\mathcal{X}_T := \{X_T : \; X \in \mathcal{X}\}$. In this setting, the elements of $\mathcal{X}$ are interpreted as admissible wealth processes starting from zero initial capital; the elements of $\mathcal{X}^x$ are called x-admissible.

Remark A.1 ("Standard" model) In the typical example, a d-dimensional semimartingale S is given and $\mathcal{X}^1$ is the set of stochastic integrals $H \cdot S$, where H is S-integrable and $H \cdot S \geq -1$. Though our main result deals with the standard model, it is natural to discuss the basic definitions and their relations with concepts of arbitrage theory in a more general framework.

Define the set of strictly 1-admissible processes $\mathcal{X}^1_> \subseteq \mathcal{X}^1$ as composed of those $X \in \mathcal{X}^1$ such that $X > -1$ and $X_- > -1$. The sets $x + \mathcal{X}^x$, $x + \mathcal{X}^x_>$, and so on, $x \in \mathbb{R}_+$, have obvious interpretations. We are particularly interested in the set $1 + \mathcal{X}^1_>$. Its elements are strictly positive wealth processes starting with unit initial capital and may be thought of as *tradable numéraires*.

For $\xi \in L^0_+$, define the *superhedging value* $x(\xi) := \inf\{x : \; \xi \in x + \mathcal{X}^x_T - L^0_+\}$. We say that the wealth-process family $\mathcal{X}$ satisfies the condition NA_1 (*no arbitrage of the first kind*) if $x(\xi) > 0$ for every $\xi \in L^0_+ \setminus \{0\}$. Alternatively, the condition NA_1 can be defined via

$$\left(\bigcap_{x>0} \{x + \mathcal{X}^x_T - L^0_+\} \right) \cap L^0_+ = \{0\}.$$

The family $\mathcal{X}$ is said to satisfy the condition NAA_1 (*no asymptotic arbitrage of the first kind*) if for any sequence $(x^n)_n$ of positive numbers with $x^n \downarrow 0$ and any sequence of value processes $X^n \in \mathcal{X}$ such that $x^n + X^n \geq 0$, we have

$$\limsup_{n\to\infty} P[x^n + X^n_T \geq 1] = 0.$$

Finally, we say that the family $\mathcal{X}$ satisfies the condition NUPBR (*no unbounded profit with bounded risk*) if the set $\{X_T : \; X \in \mathcal{X}^1_>\}$ is bounded in L^0. Since we have

$(1/2)\mathcal{X}_T^1 = \mathcal{X}_T^{1/2} \subseteq \{X_T :\ X \in \mathcal{X}_>^1\}$, the set $\{X_T :\ X \in \mathcal{X}_>^1\}$ is bounded in L^0 if and only if the set $\mathcal{X}_T^1$ is bounded in L^0.

The next result shows that all three market viability notions introduced above coincide.

Lemma A.1 *$NAA_1 \Longleftrightarrow NUPBR \Longleftrightarrow NA_1$.*

Proof $\mathrm{NAA}_1 \Rightarrow$ NUPBR: If the family $\{X_T :\ X \in \mathcal{X}_>^1\}$ fails to be bounded in L^0, then $P[1+\tilde{X}_T^n \ge n] \ge \varepsilon > 0$ for a sequence $(\tilde{X}^n) \subseteq \mathcal{X}_>^1$, and we obtain a violation of NAA_1 with $n^{-1}+n^{-1}\tilde{X}_T^n$.

NUPBR $\Rightarrow \mathrm{NA}_1$: If NA_1 fails, then there exist $\xi \in L_+^0 \setminus \{0\}$ and a sequence (X^n) with $X^n \in \mathcal{X}^{1/n}$ such that $1/n + X^n \ge \xi$. Then the sequence $nX_T^n \in \mathcal{X}^1$ fails to be bounded in L^0, in violation of NUPBR.

$\mathrm{NA}_1 \Rightarrow \mathrm{NAA}_1$: If the implication fails, then there are sequences $x^n \downarrow 0$ and $X^n \ge -x^n$ such that $P[x^n + X_T^n \ge 1] \ge 2\varepsilon > 0$. By the von Weizsäcker theorem (see [18] or [11, Theorem A.2.3]), any sequence of random variables bounded from below contains a subsequence converging in the Cesarò sense a.s. as well as all its further subsequences. We may assume without loss of generality that for $\xi^n := x^n + X_T^n$, the sequence $\bar{\xi}^n := (1/n)\sum_{i=1}^n \xi_i$ converges to $\xi \in L_+^0$. Note that $\xi \ne 0$. Indeed,

$$\begin{aligned}\varepsilon(1 - P[\bar{\xi}^n \ge \varepsilon]) &\ge \frac{1}{n}\sum_{i=1}^n E\xi^i I_{\{\bar{\xi}^n < \varepsilon\}} \ge \frac{1}{n}\sum_{i=1}^n E\xi^i I_{\{\xi^i \ge 1,\ \bar{\xi}^n < \varepsilon\}} \\ &\ge \frac{1}{n}\sum_{i=1}^n P[\xi^i \ge 1,\ \bar{\xi}^n < \varepsilon] \ge \frac{1}{n}\sum_{i=1}^n (P[\xi^i \ge 1] - P[\bar{\xi}^n \ge \varepsilon]) \\ &\ge 2\varepsilon - P[\bar{\xi}^n \ge \varepsilon].\end{aligned}$$

It follows that $P[\bar{\xi}^n \ge \varepsilon] \ge \varepsilon/(1-\varepsilon)$. Thus,

$$E[\xi \wedge 1] = \lim_n E[\bar{\xi}^n \wedge 1] \ge \varepsilon^2/(1-\varepsilon) > 0.$$

It follows that there exists $a > 0$ such that $P[\xi \ge 2a] > 0$. In view of Egorov's theorem, we can find a measurable set $\Gamma \subseteq \{\xi \ge a\}$ with $P[\Gamma] > 0$ on which $x^n + X^n \ge a$ for all sufficiently large n. But this means that the random variable $aI_\Gamma \ne 0$ can be superreplicated starting with arbitrarily small initial capital, in contradiction with the assumed condition NA_1. □

Remark A.2 (On terminology and bibliography) The conditions NAA_1 and NA_1 have clear financial meanings, whereas the boundedness in L^0 of the set $\mathcal{X}_T^1$, at first glance, looks like a technical condition—see [5]. The concept of NAA_1 first appeared in [10] in a much more general context of large financial markets, along with another fundamental notion NAA_2 (no asymptotic arbitrage of the second kind). The boundedness in L^0 of $\mathcal{X}_T^1$ was discussed in [9] (as the BK-property), in the framework of a model given by value processes; however, it was overlooked that it coincides with NAA_1 for the "stationary" model, that is, when the stochastic basis and the price process do not depend on n. The same condition appeared under the acronym NUPBR in [12] and was shown to be equivalent to NA_1 in [13].

A.2 $\mathbf{NA_1}$ and NFLVR

Remaining in the framework of the abstract model of the previous subsection, we provide here results on the relation of the condition NA_1 with other fundamental notions of arbitrage theory; compare with [9].

Define the convex sets $C := (\mathcal{X}_T - L^0_+) \cap L^\infty$ and denote by $\bar{C}$ and $\bar{C}^*$ the norm-closure and weak* closure of C in L^∞, respectively. The conditions NA, NFLVR, and NFL are defined correspondingly via

$$C \cap L^\infty_+ = \{0\}, \qquad \bar{C} \cap L^\infty_+ = \{0\}, \qquad \bar{C}^* \cap L^\infty_+ = \{0\}.$$

Consecutive inclusions induce a hierarchy of these properties via

$$\begin{array}{ccccc} C & \subseteq & \bar{C} & \subseteq & \bar{C}^*, \\ \mathit{NA} & \Longleftarrow & \mathit{NFLVR} & \Longleftarrow & \mathit{NFL}. \end{array}$$

Lemma A.2 *NFLVR* $\Longrightarrow$ *NA&NA*$_1$.

Proof Assume that NFLVR holds. Condition NA follows trivially. If NA_1 fails, then there exists a $[0,1]$-valued $\xi \in L^0_+ \setminus \{0\}$ such that for each $n \ge 1$, we can find $X^n \in \mathcal{X}^{1/n}$ with $1/n + X^n_T \ge \xi$. Then the random variables $X^n_T \wedge \xi$ belong to C and converge uniformly to ξ, contradicting NFLVR. □

To obtain the converse implication in Lemma A.2, we need an extra property. We call a model *natural* if the elements of $\mathcal{X}$ are adapted processes and for any $X \in \mathcal{X}$, $s \in [0,T)$ and $\Gamma \in \mathcal{F}_s$, the process $\tilde{X} := I_{\Gamma \cap \{X_s \le 0\}} I_{[s,T]}(X - X_s)$ is an element of $\mathcal{X}$. In words, a model is natural if an investor deciding to start trading at time s when the event Γ happened can use from this time, if $X_s \le 0$, an investment strategy that leads to a value process with the same increments as X.

Lemma A.3 *Suppose that the model is natural. If NA holds, then any $X \in \mathcal{X}$ admits the bound $X \ge -\lambda$, where $\lambda = \|X^-_T\|_\infty$.*

Proof If $P[X_s < -\lambda] > 0$, then $\tilde{X} := I_{\{X_s < -\lambda\}} I_{[s,T]}(X - X_s)$ belongs to $\mathcal{X}$ and satisfies $\tilde{X}_T \ge 0$ and $P[\tilde{X}_T > 0] > 0$, in violation of NA. □

Proposition A.4 *Suppose that the model is natural and in addition that for every $n \ge 1$ and $X \in \mathcal{X}$ with $X \ge -1/n$, the process nX is in $\mathcal{X}^1$. Then*

$$\mathit{NFLVR} \iff \mathit{NA}\ \&\ \mathit{NA}_1.$$

Proof By Lemma A.2, we only have to show the implication "$\Leftarrow$". If NFLVR fails, then there are $\xi_n \in C$ and $\xi \in L^\infty_+ \setminus \{0\}$ such that $\|\xi_n - \xi\|_\infty \le n^{-1}$. By definition, we have $\xi_n \le \eta_n = X^n_T$, where $X^n \in \mathcal{X}$. Obviously, $\|\eta^-_n\|_\infty \le n^{-1}$, and since NA holds,

$nX^n \in \mathcal{X}^1$ by virtue of Lemma A.3 and our hypothesis. By the von Weizsäcker theorem, we may assume that $\eta_n \to \eta$ a.s. Since $P[\eta > 0] > 0$, the sequence $nX_T^n \in \mathcal{X}_T^1$ tends to infinity with strictly positive probability, violating condition NUPBR or, equivalently, NA_1. □

Examples showing that the conditions NFLVR, NA, and NA_1 are all different, even for the standard model (satisfying, of course, the hypotheses of the above proposition) can be found in [7].

Assume now that $\mathcal{X}^1$ is a subset of the space $\mathcal{S}$ of semimartingales, equipped with the Emery topology given by the quasi-norm

$$\mathbf{D}(X) := \sup\{E[1 \wedge |H \cdot X_T|] : \ H \text{ is predictable}, \ |H| \le 1\}.$$

Define the condition ESM as the existence of a probability $\tilde{P} \sim P$ such that $\tilde{E}X_T \le 0$ for all processes $X \in \mathcal{X}$. A probability $\tilde{P}$ with this property is referred to as an *equivalent separating measure*. According to the Kreps–Yan separation theorem [11, Theorem 2.1.4], the conditions NFL and ESM are equivalent. The next result is proved in [9] on the basis of the paper [5], where this theorem was established for the "standard" model; see also [3].

Theorem A.3 *Suppose that $\mathcal{X}^1$ is closed in $\mathcal{S}$ and that the following concatenation property holds: for any $X, X' \in \mathcal{X}^1$ and any bounded predictable processes $H, G \ge 0$ such that $HG = 0$, the process $\tilde{X} := H \cdot X + G \cdot X'$ belongs to $\mathcal{X}^1$ if it satisfies the inequality $\tilde{X} \ge -1$. Then, under the condition NFLVR, it holds that $C = \bar{C}^*$, and as a corollary, we have*

$$\mathit{NFLVR} \iff \mathit{NFL} \iff \mathit{ESM}.$$

Remark A.4 It is shown in [15, Theorem 1.7] that the condition NA_1 is equivalent to the existence of the (unique) supermartingale numéraire in a setting where the wealth-process sets are abstractly defined via a requirement of predictable convexity (also called fork-convexity).

In the case of the "standard" model with a finite-dimensional semimartingale S describing the prices of the basic risky securities, we have the following: If S is bounded (resp. locally bounded), a separating measure is a martingale measure (resp. local martingale measure). Without any local boundedness assumption on S, we have the following result from [6], a short proof of which is given in [9] and which we use here.

Theorem A.5 *In any neighbourhood in total variation of a separating measure, there exists an equivalent probability under which S is a σ-martingale.*

It follows that if NFLVR holds, then the process S is a σ-martingale with respect to some probability measure $P' \sim P$ with density process Z'. Therefore, for any process $X = H \cdot S$ from $\mathcal{X}^1$, the process $1 + X$ is a local martingale with respect to P', or equivalently, $Z'(1 + X)$ is a local martingale with respect to P; therefore, Z' is a local martingale deflator.

Remark A.6 A counterexample in [4, Sect. 6] involving a simple one-step model shows that Theorem A.5 is not valid in markets with countably many assets. As a corollary, the condition NFLVR (equivalent in this one-step model to NA_1) is not sufficient to ensure the existence of a local martingale measure or a local martingale deflator.

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