

The numéraire portfolio in semimartingale financial models

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Received: 30 May 2006 / Accepted: 2 April 2007 / Published online: 8 August 2007
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Abstract We study the existence of the *numéraire portfolio* under predictable convex constraints in a general semimartingale model of a financial market. The numéraire portfolio generates a wealth process, with respect to which the relative wealth processes of all other portfolios are supermartingales. Necessary and sufficient conditions for the existence of the numéraire portfolio are obtained in terms of the triplet of predictable characteristics of the asset price process. This characterization is then used to obtain further necessary and sufficient conditions, in terms of a no-free-lunch-type notion. In particular, the full strength of the “No Free Lunch with Vanishing Risk” (NFLVR) condition is not needed, only the weaker “No Unbounded Profit with Bounded Risk” (NUPBR) condition that involves the boundedness in probability of the terminal values of wealth processes. We show that this notion is the minimal a-priori assumption required in order to proceed with utility optimization. The fact that it is expressed entirely in terms of predictable characteristics makes it easy to check, something that the stronger NFLVR condition lacks.

Keywords Numéraire portfolio · Semimartingale · Predictable characteristics · Free lunch · Supermartingale deflator · Log-utility

Mathematics Subject Classification (2000) 60H05 · 60H30 · 91B28

JEL Classification G11

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1 Introduction

1.1 Background and discussion of results

A broad class of models, that have been used extensively in stochastic finance, are those for which the price processes of financial instruments are considered to evolve as semimartingales. The concept of a semimartingale is very intuitive: it connotes a process that can be decomposed into a *finite variation* term that represents the “signal”, and a *local martingale* term that represents the “noise”. The reasons for the ubiquitousness of semimartingales in modeling financial asset prices are by now pretty well understood—see, for example, Delbaen and Schachermayer [10], where it is shown that restricting ourselves to the realm of locally bounded stock prices and agreeing that we should banish arbitrage by use of simple “buy-and-hold” strategies, the price process *has* to be a semimartingale. Discrete-time models can be embedded in this class, as can processes with independent increments and many other Markov processes, such as solutions to stochastic differential equations. Models that are not encompassed, but have received attention, include price processes driven by fractional Brownian motion.

In this paper we consider a general semimartingale model and make no further mathematical assumptions. On the economic side, we assume that assets have their prices determined exogenously and can be traded without “frictions”: transaction costs are non-existent or negligible. Our main concern will be a problem of dynamic stochastic optimization: to find a trading strategy whose wealth appears “better” when compared to the wealth generated by any other strategy, in the sense that the ratio of the two processes is a supermartingale. If such a strategy exists, it is essentially unique and is called *numéraire portfolio*. Necessary and sufficient conditions for the numéraire portfolio to exist are derived, in terms of the triplet of *predictable characteristics* of the stock-price returns.

Sufficient conditions for the existence of the numéraire portfolio are established in [18], who focus on the (almost equivalent) problem of maximizing expected logarithmic utility. These authors show that their conditions are also necessary, under the following assumptions: the problem of maximizing the expected log-utility from terminal wealth has a finite value, no constraints are enforced on strategies, and NFLVR holds. Becherer [4] also discusses how, under these assumptions, the numéraire portfolio exists and coincides with the log-optimal one. In both these papers, deep results of Kramkov and Schachermayer [30] on utility maximization are invoked.

Here we follow a bare-hands approach which enables us to obtain stronger results. Firstly, the assumption of finite expected log-utility is dropped entirely; there should be no reason for it, anyhow, since we are not working on the problem of log-utility optimization. Secondly, general closed convex constraints on portfolio choice can be enforced, as long as they unfold in a predictable manner. Thirdly, and perhaps most controversially, we drop the NFLVR assumption: *no* normative assumption is imposed on the model. It turns out that the numéraire portfolio can exist even when the classical *No Arbitrage* (NA) condition fails.

In the context of stochastic portfolio theory, we feel there is no need for no-free-lunch assumptions to begin with: the rôle of optimization should be to find and utilize

arbitrage opportunities in the market, rather than ban the model. It is actually possible that the optimal strategy of an investor is *not* an arbitrage (an example involves the notorious three-dimensional Bessel process and can be found in Sect. 4.3.3 of the present paper). The usual practice of assuming that we can invest unconditionally on arbitrages breaks down because of credit limit constraints: arbitrages are sure to generate, at a fixed future date, more capital than initially invested; but they can do pretty badly in the meantime, and this imposes an upper bound on the capital that can be invested. There exists an even more severe problem when trying to argue that arbitrages should be banned: in very general semimartingale financial markets *there does not seem to exist any computationally feasible way of deciding whether arbitrages exist or not*. This goes hand-in-hand with the fact that existence of equivalent martingale measures—its remarkable theoretical importance notwithstanding—is *not easy to check*, at least by looking directly at the dynamics of the stock-price process.

Our second main result comes hopefully to shed some light on this situation. Having made no model assumptions when initially trying to decide whether the numéraire portfolio exists, we now take a step backwards and in the opposite-than-usual direction: we ask ourselves *what the existence of the numéraire portfolio can tell us about free-lunch-like opportunities in the market*. Here the necessary and sufficient condition for existence of the numéraire portfolio is the boundedness in probability of the collection of terminal wealths attainable by trading (“no unbounded profit with bounded risk”, NUPBR, for short). This is one of the two conditions that comprise NFLVR; what remains, of course, is the NA condition. In the spirit of the Fundamental Theorem of Asset Pricing, we show that another mathematical equivalence to the NUPBR condition is the *existence of equivalent supermartingale deflators*, a concept closely related but strictly weaker than equivalent martingale measures. A similar result appears in [8], where the results of Kramkov and Schachermayer [30] are again used.

We then go on further and ask how severe this NUPBR assumption really is. The answer is simple: when this condition fails, one cannot do utility optimization for *any* utility function; conversely, if this assumption holds, one can proceed with utility maximization as usual. The main advantage of not assuming the full NFLVR condition is that *there is a direct way of checking the validity of the weaker NUPBR condition in terms of the predictable characteristics of the price process*. No such characterization exists for the NA condition, as Example 4.7 in Sect. 4.3 demonstrates. Furthermore, our result can be used to understand the gap between the concepts of NA and the stronger NFLVR; *the existence of the numéraire portfolio is exactly the bridge needed to take us from NA to NFLVR*. This was known for continuous-path processes since the paper [11] of Delbaen and Schachermayer; here we do it for the general case.

1.2 Synopsis

After this short subsection, in the remainder of this section we recall probabilistic concepts to be used throughout.

Section 2 introduces the financial market model, the ways in which agents can invest in this market, and the constraints they face. In Sect. 3 we introduce the

numéraire portfolio. We discuss how it relates to other notions and conclude with our main Theorem 3.15 that provides necessary and sufficient conditions for the existence of the numéraire portfolio in terms of the predictable characteristics of the stock-price processes. Section 4 deals with the connections between the numéraire portfolio and free lunches. The main result there is Theorem 4.12, which can be seen as another version of the Fundamental Theorem of Asset Pricing.

Certain proofs that are not given in Sects. 3 and 4 occupy the next four sections. In the self-contained Sect. 5 we describe necessary and sufficient conditions for the existence of wealth processes that are increasing and not constant. In Sect. 6 we prove our main Theorem 3.15. Section 7 contains a result on rates of convergence to zero of positive supermartingales, which is used to study an asymptotic optimality property of the numéraire portfolio. Finally, Sect. 8 completes proving our second main Theorem 4.12.

In order to stay as self-contained as possible, Appendices are included on: (a) measurable random subsets and selections; (b) semimartingales up to infinity and the corresponding “stochastic integration up to infinity”; and (c) σ -localization.

1.3 Remarks of probabilistic nature

For results concerning the general theory of stochastic processes, we refer the reader to the book [21] of Jacod and Shiryaev, especially the first two chapters.

We are given a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$, where the filtration $\mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}$ is assumed to satisfy the usual hypotheses of right-continuity and augmentation by the \mathbb{P} -null sets. The probability measure \mathbb{P} will be fixed throughout and every formula, relationship, etc., is supposed to be valid \mathbb{P} -almost surely (\mathbb{P} -a.s.)

The *predictable σ -algebra* on the base space $\Omega \times \mathbb{R}_+$ will be denoted by \mathcal{P} —if π is a d -dimensional predictable process we write $\pi \in \mathcal{P}(\mathbb{R}^d)$. For any adapted, right-continuous process Y that admits left-hand limits, we denote by Y_- its *predictable left-continuous version* and its *jump process* is $\Delta Y := Y - Y_-$.

For a d -dimensional semimartingale X and $\pi \in \mathcal{P}(\mathbb{R}^d)$, we denote by $\pi \cdot X$ the *stochastic integral* process, whenever this makes sense, in which case we shall be referring to π as being *X -integrable*. We are assuming *vector stochastic integration*, good accounts of which can be found in [5, 6, 21]. For two semimartingales X and Y , $[X, Y] := XY - X_- \cdot Y - Y_- \cdot X$ is their *quadratic covariation* process.

The *stochastic exponential* $\mathcal{E}(Y)$ of the scalar semimartingale Y , null at 0, is the unique solution Z of the stochastic integral equation $Z = 1 + Z_- \cdot Y$ and is given by

$$\mathcal{E}(Y) = \exp \left\{ Y - \frac{1}{2} [Y^c, Y^c] \right\} \cdot \prod_{s \leq \cdot} \{ (1 + \Delta Y_s) \exp(-\Delta Y_s) \}, \tag{1.1}$$

where Y^c denotes the *continuous martingale part* of the semimartingale Y . The stochastic exponential $Z = \mathcal{E}(Y)$ satisfies $Z > 0$ and $Z_- > 0$ if and only if $\Delta Y > -1$. Given a semimartingale Z which satisfies $Z > 0$ and $Z_- > 0$, we can invert the stochastic exponential operator and get the *stochastic logarithm* $\mathcal{L}(Z)$, which is defined as $\mathcal{L}(Z) := (1/Z_-) \cdot Z$ and satisfies $\Delta \mathcal{L}(Z) > -1$.

2 The market, investments, and constraints

2.1 The asset-prices model

On the given stochastic basis $(\Omega, \mathcal{F}, \mathbf{F}, \mathbb{P})$ we consider d strictly positive semimartingales S^1, \dots, S^d that model the prices of d assets; we shall refer to these as *stocks*. There is also another process S^0 , representing the *money market* or *bank account*—this asset is a “benchmark”, in the sense that wealth processes will be quoted in units of S^0 and not nominally. As is usually done in this field, we assume $S^0 \equiv 1$, making S^1, \dots, S^d already *discounted* asset prices. This does not affect the generality of the discussion, since, otherwise, we can divide all $S^i, i = 0, 1, \dots, d$, by S^0 .

For all $i = 1, \dots, d$, S^i and S^i_- are assumed strictly positive; therefore, there exists a d -dimensional semimartingale $X \equiv (X^1, \dots, X^d)$ with $X_0 = 0, \Delta X^i > -1$ and $S^i = S^0 \mathcal{E}(X^i)$ for $i = 1, \dots, d$. We interpret X as the *discounted returns* that generate the asset prices S in a multiplicative way. In our discussion we shall be using the returns process X , not the stock-price process S directly.

Our financial planning horizon will be $\llbracket 0, T \rrbracket := \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid t \leq T(\omega)\}$ where T is a *possibly infinite-valued stopping time*. Observe that, as usual, even if T takes infinite values, the time-point at infinity is *not* included in the definition of $\llbracket 0, T \rrbracket$. All processes will be considered as being constant and equal to their value at T for all times after T , i.e., every process Z is equal to the *stopped process* at time T , defined via $Z_t^T := Z_{t \wedge T}$ for all $t \in \mathbb{R}_+$. We can assume further, without loss of generality, that \mathcal{F}_0 is \mathbb{P} -trivial (thus, all \mathcal{F}_0 -measurable random variables are constants) and that $\mathcal{F} = \mathcal{F}_T := \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_{t \wedge T}$.

Remark 2.1 Under our model we have $S^i > 0$ and $S^i_- > 0$; to be in par with the papers [10, 14] on no-free-lunch criteria, we should allow for models with possibly negative asset prices (for example, forward contracts). All our subsequent work carries over to these models. We choose to work in the above set-up because it is somehow more intuitive and applicable: almost every model used in practice is written in this way. A follow-up to this discussion is Sect. 4.8.

The predictable characteristics of the returns process X will be very important in our discussion. To this end, we fix the *canonical truncation function* $x \mapsto x \mathbb{I}_{\{|x| \leq 1\}}$ (we use \mathbb{I}_A to denote the indicator function of some set A) and write the *canonical representation* of the semimartingale X , namely

$$X = X^c + B + [x \mathbb{I}_{\{|x| \leq 1\}}] * (\mu - \eta) + [x \mathbb{I}_{\{|x| > 1\}}] * \mu. \tag{2.1}$$

Some remarks are in order. Here μ is the *jump measure* of X , i.e., the random counting measure on $\mathbb{R}_+ \times \mathbb{R}^d$ defined by

$$\mu(\llbracket 0, t \rrbracket \times A) := \sum_{0 < s \leq t} \mathbb{I}_{A \setminus \{0\}}(\Delta X_s), \quad \text{for } t \in \mathbb{R}_+ \text{ and } A \subseteq \mathbb{R}^d. \tag{2.2}$$

Thus, the last process in (2.1) is just $[x \mathbb{I}_{\{|x| > 1\}}] * \mu \equiv \sum_{0 < s \leq \cdot} \Delta X_s \mathbb{I}_{\{|\Delta X_s| > 1\}}$, the *sum of the “big” jumps of X* ; throughout the paper, the asterisk denotes integration with

respect to random measures. Once this term is subtracted from X , what remains is a semimartingale with bounded jumps; thus, a *special* semimartingale. This, in turn, can be decomposed uniquely into a *predictable finite variation* part, denoted by B in (2.1), and a *local martingale* part. Finally, this last local martingale part can be decomposed further: into its *continuous* part, denoted by X^c in (2.1); and its *purely discontinuous* part, identified as the local martingale $[x\mathbb{I}_{\{|x|\leq 1\}}] * (\mu - \eta)$. Here η is the *predictable compensator* of the measure μ ; so, the purely discontinuous part is just a *compensated sum of “small” jumps*.

We introduce the *quadratic covariation* process $C := [X^c, X^c]$ of X^c , call (B, C, η) the *triplet of predictable characteristics* of X , and define the predictable increasing scalar process $G := \sum_{i=1}^d (C^{i,i} + \text{Var}(B^i) + [1 \wedge |x|^2] * \eta)$. Then all three B, C , and η are absolutely continuous with respect to G , thus,

$$B = b \cdot G, \quad C = c \cdot G, \quad \text{and} \quad \eta = G \otimes \nu. \tag{2.3}$$

Here b, c , and ν are predictable; b is a vector process, c a nonnegative-definite matrix-valued process, and ν a process with values in the set of Lévy measures; the symbol “ \otimes ” denotes product measure. Note that any \tilde{G} with $d\tilde{G}_t \sim dG_t$ can be used in place of G ; the actual choice of an increasing process G reflects the notion of *operational clock* (as opposed to the natural time flow, described by t). In an abuse of terminology, we shall refer to (b, c, ν) also as the *triplet of predictable characteristics of X* ; this depends on G , but the validity of all results does not.

Remark 2.2 The properties of c being a symmetric nonnegative-definite process and ν a Lévy-measure-valued process in general hold $\mathbb{P} \otimes G$ -a.e. We shall assume that they hold *everywhere* on $[[0, T]]$; we can always do this by altering them on a predictable set of $\mathbb{P} \otimes G$ -measure zero to $c \equiv 0$ and $\nu \equiv 0$ (see [21], Proposition 2.9).

Remark 2.3 If X is *quasi-left-continuous* (i.e., if no jumps occur at predictable times), G is continuous; but if we want to include discrete-time models in our discussion, we must allow for G to have jumps. Since C is continuous and (2.1) gives $\mathbb{E}[\Delta X_\tau \mathbb{I}_{\{|\Delta X_\tau| \leq 1\}} \mid \mathcal{F}_{\tau-}] = \Delta B_\tau$ for every predictable time τ , we get

$$c = 0 \quad \text{and} \quad b = \int x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx) \quad \text{on the predictable set } \{\Delta G > 0\}. \tag{2.4}$$

The following concept of *drift rate* will be used throughout the paper.

Definition 2.4 Let X be any semimartingale with canonical representation (2.1), and consider the process G such that (2.3) holds. On $\{\int |x| \mathbb{I}_{\{|x| > 1\}} \nu(dx) < \infty\}$, the *drift rate* (with respect to G) of X is defined as $b + \int x \mathbb{I}_{\{|x| > 1\}} \nu(dx)$.

The range of definition, $\{\int |x| \mathbb{I}_{\{|x| > 1\}} \nu(dx) < \infty\}$, for the drift rate does not depend on the choice of operational clock G , though the drift rate itself does. Whenever the increasing process $[|x| \mathbb{I}_{\{|x| > 1\}}] * \eta = (\int |x| \mathbb{I}_{\{|x| > 1\}} \nu(dx)) \cdot G$ is *finite* (this happens if and only if X is a *special* semimartingale), the predictable process

$$B + [x \mathbb{I}_{\{|x| > 1\}}] * \eta = \left(b + \int x \mathbb{I}_{\{|x| > 1\}} \nu(dx) \right) \cdot G$$

is called the *drift process* of X . If drift processes exist, drift rates exist too; the converse is not true. Semimartingales that are *not* special might have well-defined drift rates; for instance, a σ -martingale is a semimartingale with drift rate identically equal to zero. See Appendix 3 on σ -localization for further discussion.

2.2 Portfolios and wealth processes

A financial agent starts with some positive initial capital, which we normalize to $W_0 = 1$, and can invest in stocks by choosing a *portfolio* represented by a predictable, d -dimensional and X -integrable process π . With π_t^i representing the *proportion of current wealth* invested in stock i at time t , $\pi_t^0 := 1 - \sum_{i=1}^d \pi_t^i$ is the proportion invested in the money market.

Some restrictions have to be enforced, so that the agent cannot use so-called *doubling strategies*. The assumption prevailing in this context is that the wealth process should be uniformly bounded from below by some constant—a *credit limit* that the agent faces. We shall set this credit limit at zero (uniformly over strategies); one can regard this as shifting the wealth process to some constant, and working with this relative credit line instead of the absolute one.

The above discussion leads to the following definition: a wealth process will be called *admissible*, if it and its left-continuous version stay strictly positive. Let us denote the *discounted* wealth process generated from a portfolio π by W^π ; we must have $W^\pi > 0$ and $W_-^\pi > 0$, as well as

$$\frac{dW_t^\pi}{W_{t-}^\pi} = \sum_{i=0}^d \pi_t^i \frac{dS_t^i}{S_{t-}^i} = \sum_{i=1}^d \pi_t^i dX_t^i = \pi_t^\top dX_t, \quad \text{equivalently, } W^\pi = \mathcal{E}(\pi \cdot X). \tag{2.5}$$

2.3 Further constraints on portfolios

We start with an example in order to motivate Definition 2.6 below.

Example 2.5 Suppose that the agent is prevented from selling stock short. This means $\pi^i \geq 0$ for all $i = 1, \dots, d$, or that $\pi(\omega, t) \in (\mathbb{R}_+)^d$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$. If we further prohibit borrowing from the money market then also $\pi^0 \geq 0$; setting $\mathcal{C} := \{p \in \mathbb{R}^d \mid p^i \geq 0 \text{ and } \sum_{i=1}^d p^i \leq 1\}$, the prohibition of short sales and borrowing translates into the requirement $\pi(\omega, t) \in \mathcal{C}$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$.

The example leads us to consider all possible constraints that can arise this way; although in the above particular case the set \mathcal{C} was non-random, we shall soon encounter situations where the constraints depend on both time and the path.

Definition 2.6 Consider a set-valued process $\mathcal{C} : \llbracket 0, T \rrbracket \rightarrow \mathcal{B}(\mathbb{R}^d)$, where $\mathcal{B}(\mathbb{R}^d)$ is the Borel σ -algebra on \mathbb{R}^d . A $\pi \in \mathcal{P}(\mathbb{R}^d)$ will be called \mathcal{C} -constrained if $\pi(\omega, t) \in \mathcal{C}(\omega, t)$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$. We denote by $\Pi_{\mathcal{C}}$ the class of all \mathcal{C} -constrained, predictable, and X -integrable processes that satisfy $\pi^\top \Delta X > -1$, i.e., are such that W^π is admissible.

The requirement $\pi^\top \Delta X > -1$ is here to ensure that we can define the admissible wealth process W^π , i.e., that the wealth will remain strictly positive. Let us use this requirement to give other constraints of this type. Since these actually follow from the definitions, they will not constrain the wealth processes further; the point is that we can always include them in our constraint set.

Example 2.7 (Natural constraints) An admissible strategy generates a wealth process that starts positive and stays positive. Thus, if $W^\pi = \mathcal{E}(\pi \cdot X)$ then we have $\Delta W^\pi \geq -W^\pi$, or $\pi^\top \Delta X \geq -1$. (Of course, the inequalities are actually strict, but we use this weaker version to keep the set \mathcal{C}_0 , appearing below, closed.) Recalling the definition of the random measure ν from (2.3), we see that an equivalent requirement is

$$\nu[\pi^\top x < -1] \equiv \nu[\{x \in \mathbb{R}^d \mid \pi^\top x < -1\}] = 0, \quad \mathbb{P} \otimes G\text{-almost everywhere.}$$

Define now *the random set-valued process of natural constraints*

$$\mathcal{C}_0 := \{p \in \mathbb{R}^d \mid \nu[p^\top x < -1] = 0\} \tag{2.6}$$

(randomness comes through ν). Since $\pi^\top \Delta X > -1$, $\pi \in \Pi_{\mathcal{C}}$ implies $\pi \in \Pi_{\mathcal{C} \cap \mathcal{C}_0}$.

Note that \mathcal{C}_0 is not deterministic in general—random constraints are not introduced just for the sake of generality, but because they arise naturally in portfolio choice settings. In Sect. 3.3 we shall impose more structure on the set-valued process \mathcal{C} : *convexity, closedness, and predictability*. The above Examples 2.5 and 2.7 have these properties; the “predictability structure” should be clear for \mathcal{C}_0 , which involves the predictable process ν .

3 The numéraire portfolio: definitions, general discussion, and predictable characterization

3.1 The numéraire portfolio

The following is a central notion of the paper.

Definition 3.1 A process $\rho \in \Pi_{\mathcal{C}}$ will be called *numéraire portfolio* if for every $\pi \in \Pi_{\mathcal{C}}$ the *relative wealth process* W^π / W^ρ is a supermartingale.

The term “numéraire portfolio” was introduced by Long [33]; he defined it as a portfolio ρ that makes W^π / W^ρ a *martingale* for every portfolio π , then went on to show that this requirement is equivalent, under some additional assumptions, to the absence of arbitrage for discrete-time and Itô-process models. Some authors give the numéraire portfolio other names as *growth optimal* and *benchmark* (see, for example, Platen [35] who uses the “numéraire” property as an approach to derivatives pricing, portfolio optimization, etc.). Definition 3.1 in this form first appeared in [4], where we send the reader for the history of this concept. An observation from that paper is that the wealth process generated by numéraire portfolios is unique: if there are

two numéraire portfolios ρ_1 and ρ_2 in $\Pi_{\mathcal{E}}$ then both W^{ρ_1}/W^{ρ_2} and W^{ρ_2}/W^{ρ_1} are supermartingales and Jensen’s inequality shows that they are equal.

Observe that W_T^ρ is always well-defined, even on $\{T = \infty\}$, since $1/W^\rho$ is a positive supermartingale and the supermartingale convergence theorem implies that W_T^ρ exists, though it might take the value $+\infty$ on $\{T = \infty\}$. A condition of the form $W_T^\rho < +\infty$ will be essential when we consider free lunches in Sect. 4.

Remark 3.2 The numéraire portfolio was introduced in Definition 3.1 as the solution to some sort of optimization problem. It has at least four more optimality properties that we now mention; these have already been noted in the literature—check the appropriate places in the paper, where they are further discussed, for references. If ρ is the numéraire portfolio then:

- It maximizes the *growth rate* over all portfolios (Sect. 3.5).
- It maximizes the *asymptotic growth* of the wealth process it generates over all portfolios (Proposition 3.21).
- It solves the *relative log-utility maximization* problem (Sect. 3.7).
- $(W^\rho)^{-1}$ minimizes the *reverse relative entropy* among all *supermartingale deflators* (Sect. 4.4).

We now state the basic problem that will occupy us in this section; its solution is the content of Theorem 3.15.

Problem 3.3 Find necessary and sufficient conditions for the existence of the numéraire portfolio in terms of the triplet of predictable characteristics of the returns process X (equivalently, of the stock-price process S).

3.2 Preliminary necessary and sufficient conditions for the existence of the numéraire portfolio

To decide whether $\rho \in \Pi_{\mathcal{E}}$ is the numéraire portfolio, we must check whether W^π/W^ρ is a supermartingale for all $\pi \in \Pi_{\mathcal{E}}$, so let us derive a convenient expression for this ratio.

Consider a baseline portfolio $\rho \in \Pi_{\mathcal{E}}$ that generates a wealth W^ρ and take any other portfolio $\pi \in \Pi_{\mathcal{E}}$; their relative wealth process is given by the ratio $W^\pi/W^\rho = \mathcal{E}(\pi \cdot X)/\mathcal{E}(\rho \cdot X)$ from (2.5), which can be further expressed as follows.

Lemma 3.4 *Suppose that Y and R are two scalar semimartingales with $\Delta Y > -1$ and $\Delta R > -1$. Then $\mathcal{E}(Y)/\mathcal{E}(R) = \mathcal{E}(Z)$, where*

$$Z = Y - R - [Y^c - R^c, R^c] - \sum_{s \leq \cdot} \left\{ \Delta(Y_s - R_s) \frac{\Delta R_s}{1 + \Delta R_s} \right\}. \tag{3.1}$$

Proof The process $\mathcal{E}(R)^{-1}$ is locally bounded away from zero, so the stochastic logarithm Z of $\mathcal{E}(Y)/\mathcal{E}(R)$ exists. Furthermore, the process on the right-hand side of (3.1) is well-defined and a semimartingale, since $\sum_{s \leq \cdot} |\Delta R_s|^2 < \infty$ and $\sum_{s \leq \cdot} |\Delta Y_s \Delta R_s| < \infty$. Now, $\mathcal{E}(Y) = \mathcal{E}(R)\mathcal{E}(Z) = \mathcal{E}(R + Z + [R, Z])$, by Yor’s

formula. Taking stochastic logarithms on both sides of the last equation we get $Y = R + Z + [R, Z]$. This is an equation for the process Z ; by splitting it into continuous and purely discontinuous parts, one can guess, then verify, that it is solved by the right-hand side of (3.1). \square

Using Lemma 3.4 and (2.5) we get

$$\frac{W^\pi}{W^\rho} = \mathcal{E}((\pi - \rho) \cdot X^{(\rho)}), \quad \text{with } X^{(\rho)} := X - (c\rho) \cdot G - \left[\frac{\rho^\top x}{1 + \rho^\top x} x \right] * \mu;$$

here μ is the jump measure of X in (2.2), and G is the operational clock of (2.3).

We are interested in ensuring that W^π / W^ρ is a supermartingale. This relative wealth process is strictly positive, so the supermartingale property is equivalent to the σ -supermartingale one, which is in turn equivalent to requiring that the drift rate be finite and negative. (For drift rates, see Definition 2.4. For the σ -localization technique, see [23]; an overview of what is needed here is in Appendix 3, in particular, Propositions 11.2 and 11.3.) Since $W^\pi / W^\rho = \mathcal{E}((\pi - \rho) \cdot X^{(\rho)})$, the condition of negativity on the drift rate of W^π / W^ρ is equivalent to the requirement that the drift rate of the process $(\pi - \rho) \cdot X^{(\rho)}$ be negative. Straightforward computations show that, when it exists, this drift rate is

$$\text{rel}(\pi \mid \rho) := (\pi - \rho)^\top b - (\pi - \rho)^\top c\rho + \int \vartheta_{\pi|\rho}(x) \nu(dx). \tag{3.2}$$

(The notation $\text{rel}(\pi \mid \rho)$ stresses that this quantity is the rate of return of the *relative* wealth process W^π / W^ρ .) The integrand $\vartheta_{\pi|\rho}(\cdot)$ in (3.2) is

$$\vartheta_{\pi|\rho}(x) := \left[\frac{(\pi - \rho)^\top x}{1 + \rho^\top x} - (\pi - \rho)^\top x \mathbb{I}_{\{|x| \leq 1\}} \right] = \frac{1 + \pi^\top x}{1 + \rho^\top x} - 1 - (\pi - \rho)^\top x \mathbb{I}_{\{|x| \leq 1\}};$$

this satisfies $\nu[x \in \mathbb{R}^d \mid \vartheta_{\pi|\rho}(x) \leq -1 \text{ and } |x| > 1] = 0$, while on $\{|x| \leq 1\}$ (near $x = 0$) it behaves like $(\rho - \pi)^\top x x^\top \rho$, comparable to $|x|^2$. The integral in (3.2), therefore, always makes sense, but can take the value $+\infty$; the drift rate of W^π / W^ρ takes values in $\mathbb{R} \cup \{+\infty\}$, and the quantity of (3.2) is well-defined.

Thus, W^π / W^ρ is a supermartingale if and only if $\text{rel}(\pi \mid \rho) \leq 0$, $\mathbb{P} \otimes G$ -almost everywhere. Using this last fact we get preliminary necessary and sufficient conditions needed to solve Problem 3.3. In a different, more general form (involving also “consumption”) these have already appeared in [18].

Lemma 3.5 *Suppose that the constraints \mathfrak{C} imply the natural constraints of (2.6) (i.e., $\mathfrak{C} \subseteq \mathfrak{C}_0$), and consider a process ρ with $\rho(\omega, t) \in \mathfrak{C}(\omega, t)$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$. This ρ is the numéraire portfolio in the class $\Pi_{\mathfrak{C}}$ if and only if:*

- (1) $\text{rel}(\pi \mid \rho) \leq 0$, $\mathbb{P} \otimes G$ -a.e. for every $\pi \in \mathcal{P}(\mathbb{R}^d)$ with $\pi(\omega, t) \in \mathfrak{C}(\omega, t)$
- (2) ρ is predictable and
- (3) ρ is X -integrable

Proof The three conditions are clearly sufficient for ensuring that W^π / W^ρ is a supermartingale for all $\pi \in \Pi_{\mathcal{C}}$.

The necessity is trivial, but for the fact that condition (1) is to hold not only for all $\pi \in \Pi_{\mathcal{C}}$, but for any predictable process π (not necessarily X -integrable) such that $\pi(\omega, t) \in \mathcal{C}(\omega, t)$. Suppose condition (1) holds for all $\pi \in \Pi_{\mathcal{C}}$; first, take any $\xi \in \mathcal{P}$ such that $\xi(\omega, t) \in \mathcal{C}(\omega, t)$ and $\xi^\top \Delta X > -1$. Then $\xi_n := \xi \mathbb{I}_{\{|\xi| \leq n\}} + \rho \mathbb{I}_{\{|\xi| > n\}}$ belongs to $\Pi_{\mathcal{C}}$, so $\text{rel}(\xi \mid \rho) \mathbb{I}_{\{|\xi| \leq n\}} = \text{rel}(\xi_n \mid \rho) \mathbb{I}_{\{|\xi| \leq n\}} \leq 0$; sending n to infinity we get $\text{rel}(\xi \mid \rho) \leq 0$. Now pick any $\xi \in \mathcal{P}(\mathbb{R}^d)$ such that $\xi(\omega, t) \in \mathcal{C}(\omega, t)$; we have $\xi^\top \Delta X \geq -1$ but not necessarily $\xi^\top \Delta X > -1$. Then, for $n \in \mathbb{N}$, $\xi_n := (1 - n^{-1})\xi$ also satisfies $\xi_n \in \mathcal{P}(\mathbb{R}^d)$ and $\xi_n(\omega, t) \in \mathcal{C}(\omega, t)$, and further $\xi_n^\top \Delta X > -1$; it follows that $\text{rel}(\xi_n \mid \rho) \leq 0$. Fatou’s lemma now gives $\text{rel}(\xi \mid \rho) \leq 0$. \square

In order to solve Problem 3.3, the conditions of Lemma 3.5 will be tackled one by one. For condition (1), it will turn out that one has to solve, for each fixed $(\omega, t) \in \llbracket 0, T \rrbracket$, a convex optimization problem over the set $\mathcal{C}(\omega, t)$. It is obvious that if (1) above is to hold for \mathcal{C} then it must also hold for the closed convex hull of \mathcal{C} , so we might as well assume that \mathcal{C} is closed and convex. For condition (2), in order to prove that the solution we get is predictable, the set-valued process \mathcal{C} must have some predictable structure; we describe in the next subsection how this is done. After that, a simple test will give us condition (3), and we shall be able to provide the solution of Problem 3.3 in Theorem 3.15.

3.3 The predictable, closed convex structure of constraints

We start with a remark concerning *market degeneracies*, i.e., linear dependence that some stocks might exhibit at some points of the base space, causing seemingly different portfolios to produce the exact same wealth processes; such portfolios should then be treated as equivalent. To formulate this notion, consider two portfolios π_1 and π_2 with $W^{\pi_1} = W^{\pi_2}$. Take stochastic logarithms on both sides of the last equality to get $\pi_1 \cdot X = \pi_2 \cdot X$. Then $\zeta := \pi_2 - \pi_1$ satisfies $\zeta \cdot X \equiv 0$; this is equivalent to $\zeta \cdot X^c = 0$, $\zeta^\top \Delta X = 0$, and $\zeta \cdot B = 0$, and suggests the following definition.

Definition 3.6 For a triplet of predictable characteristics (b, c, ν) , the linear-subspace-valued process of *null investments* \mathfrak{N} is the set of vectors (depending on (ω, t) , of course) for which nothing happens if one invests in them, namely,

$$\mathfrak{N}(\omega, t) := \left\{ \zeta \in \mathbb{R}^d \mid \zeta^\top c(\omega, t) = 0, \nu(\omega, t)[\zeta^\top x \neq 0] = 0 \text{ and } \zeta^\top b(\omega, t) = 0 \right\}. \tag{3.3}$$

We have $W^{\pi_1} = W^{\pi_2}$ if and only if $\pi_2(\omega, t) - \pi_1(\omega, t) \in \mathfrak{N}(\omega, t)$, for $\mathbb{P} \otimes G$ -almost every $(\omega, t) \in \llbracket 0, T \rrbracket$; then, the portfolios π_1 and π_2 are considered identical.

Definition 3.7 The \mathbb{R}^d -set-valued process \mathcal{C} will be said to impose *predictable closed convex constraints* if:

- (1) $\mathfrak{N}(\omega, t) \subseteq \mathcal{C}(\omega, t)$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$
- (2) $\mathcal{C}(\omega, t)$ is a closed convex set, for all $(\omega, t) \in \llbracket 0, T \rrbracket$ and

- (3) \mathcal{C} is predictably measurable, in the sense that for any closed $F \subseteq \mathbb{R}^d$ we have $\{\mathcal{C} \cap F \neq \emptyset\} := \{(\omega, t) \in \llbracket 0, T \rrbracket \mid \mathcal{C}(\omega, t) \cap F \neq \emptyset\} \in \mathcal{P}$

Note the insistence that (1), (2) must hold for every $(\omega, t) \in \llbracket 0, T \rrbracket$, not just in an “almost every” sense. Requirement (1) says that we are giving investors *at least* the freedom to do nothing: if an investment is to lead to absolutely no profit or loss, one should be free to do it. In the non-degenerate case this just becomes $0 \in \mathcal{C}(\omega, t)$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$. Appendix 1 discusses further the measurability requirement (3) and its equivalence with other definitions of measurability.

The *natural constraints* \mathcal{C}_0 of (2.6) satisfy the requirements of Definition 3.7. For the predictability requirement, write $\mathcal{C}_0 = \{p \in \mathbb{R}^d \mid \int \kappa(1 + p^\top x)v(dx) = 0\}$, where $\kappa(x) := (x \wedge 0)^2 / (1 + (x \wedge 0)^2)$; then use Lemma 9.4 in conjunction with Remark 2.2, which provides a version of the characteristics such that the integrals in the above representation of \mathcal{C}_0 make sense for all $(\omega, t) \in \llbracket 0, T \rrbracket$. In view of this

we shall *always* assume $\mathcal{C} \subseteq \mathcal{C}_0$ from now on,

since, otherwise, we can replace \mathcal{C} by $\mathcal{C} \cap \mathcal{C}_0$ (and use the fact that intersections of closed predictable set-valued processes are also predictable—see Lemma 9.3 of Appendix 1).

3.4 Unbounded increasing profit

We proceed with an effort to understand condition (1) in Lemma 3.5. In this subsection we state a sufficient condition for its failure in terms of predictable characteristics. In the next subsection, when we state our first main theorem about the existence of the numéraire portfolio, we shall see that this condition is also necessary. Its failure is intimately related to the existence of wealth processes that start with unit capital, make some wealth with positive probability, and are increasing. The existence of such a possibility in a financial market amounts to the most egregious form of arbitrage.

Definition 3.8 The predictable set-valued process $\check{\mathcal{C}} := \bigcap_{a>0} a\mathcal{C}$ is the set of *cone points* (or *recession cone*) of \mathcal{C} . A portfolio $\pi \in \Pi_{\check{\mathcal{C}}}$ will be said to generate an *Unbounded Increasing Profit* (UIP), if the wealth process W^π is increasing ($\mathbb{P}[W_s^\pi \leq W_t^\pi, \forall s < t \leq T] = 1$), and if $\mathbb{P}[W_T^\pi > 1] > 0$. If no such portfolio exists, then we say that the *No Unbounded Increasing Profit* (NUIP) condition holds.

The qualifier “unbounded” stems from the fact that since $\pi \in \Pi_{\check{\mathcal{C}}}$, an agent has unconstrained leverage on the position π and can invest unconditionally; by doing so, the agent’s wealth will be multiplied accordingly. It should be clear that the numéraire portfolio cannot exist, if such strategies exist. To obtain a connection with predictable characteristics, we also give the definition of the immediate arbitrage opportunity vectors in terms of the characteristic triplet.

Definition 3.9 For a triplet of predictable characteristics (b, c, ν) , the set-valued process \mathcal{I} of *immediate arbitrage opportunities* is defined for any $(\omega, t) \in \Omega \times \mathbb{R}_+$

as the set of vectors $\xi \in \mathbb{R}^d \setminus \mathfrak{N}(\omega, t)$ for which

$$(1) \xi^\top c = 0, \quad (2) \nu[\xi^\top X < 0] = 0, \quad \text{and} \quad (3) \xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx) \geq 0.$$

(We have hidden the dependence of (b, c, ν) on (ω, t) above to ease the reading.)

\mathfrak{N} -valued processes satisfy these three conditions, but cannot be considered “arbitrage opportunities,” since they have zero returns. One can see that \mathfrak{J} is cone-valued with the whole “face” \mathfrak{N} removed.

Assume, for simplicity of discussion only, that X is a Lévy process; and that we can find a vector $\xi \in \mathfrak{J}$ (which is deterministic). In Definition 3.9, condition (1) implies that $\xi \cdot X$ has no diffusion part; (2) implies that $\xi \cdot X$ has no negative jumps; whereas, (3) turns out to imply that $\xi \cdot X$ has nonnegative drift and is of finite variation (though this is not as obvious). Using $\xi \notin \mathfrak{N}$, we get that $\xi \cdot X$ is actually non-zero and increasing, and the same will hold for $W^\xi = \mathcal{E}(\xi \cdot X)$; see Sect. 5.1 for a thorough discussion of the general (not necessarily Lévy-process) case.

Proposition 3.10 *The NUIP condition is equivalent to requiring that the predictable set $\{\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset\} := \{(\omega, t) \in \llbracket 0, T \rrbracket \mid \mathfrak{J}(\omega, t) \cap \check{\mathfrak{C}}(\omega, t) \neq \emptyset\}$ be $\mathbb{P} \otimes G$ -null.*

The proof of this result is given in Sect. 5. Section 5.1 contains one side of the argument (if an UIP exists, then $\{\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset\}$ cannot be $\mathbb{P} \otimes G$ -null) and makes a rather easy reading. The other direction, though it follows from the same idea, has a “measurable selection” flavor and the reader might wish to skim it.

Remark 3.11 We describe briefly the connection between Proposition 3.10 and our original Problem 3.3. We discuss how if $\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset$ has non-zero $\mathbb{P} \otimes G$ -measure, one cannot find a process $\rho \in \Pi_{\mathfrak{C}}$ such that $\text{rel}(\pi \mid \rho) \leq 0$ holds for all $\pi \in \Pi_{\mathfrak{C}}$. Indeed, a standard measurable selection argument (for details, the reader should check Sect. 5) allows us to infer the existence of a process ξ such that $\xi(\omega, t) \in \mathfrak{J}(\omega, t) \cap \check{\mathfrak{C}}(\omega, t)$ on $\{\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset\}$ and $\xi = 0$, otherwise. Now suppose that ρ satisfies $\text{rel}(\pi \mid \rho) \leq 0$, for all $\pi \in \Pi_{\mathfrak{C}}$. Since $\xi \in \Pi_{\check{\mathfrak{C}}}$, we have $n\xi \in \Pi_{\mathfrak{C}}$ for all $n \in \mathbb{N}$, as well as $(1 - n^{-1})\rho + \xi \in \Pi_{\mathfrak{C}}$ from convexity; but \mathfrak{C} is closed-set-valued, so $\rho + \xi \in \Pi_{\mathfrak{C}}$. Now from (3.2) and the definition of \mathfrak{J} , we see that

$$\text{rel}(\rho + \xi \mid \rho) = \dots = \xi^\top b - \int \xi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx) + \int \frac{\xi^\top x}{1 + \rho^\top x} \nu(dx) > 0$$

holds on $\{\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset\}$, which has positive $\mathbb{P} \otimes G$ -measure. This is a contradiction: there cannot exist any $\rho \in \Pi_{\mathfrak{C}}$ satisfying $\text{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi_{\mathfrak{C}}$.

Proving the converse—namely, if $\{\mathfrak{J} \cap \check{\mathfrak{C}} = \emptyset\}$ is $\mathbb{P} \otimes G$ -null then one can find a $\rho \in \Pi_{\mathfrak{C}}$ that satisfies $\text{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi_{\mathfrak{C}}$ —is more involved and will be taken on in Sect. 6.

Example 3.12 If $\nu \equiv 0$, an immediate arbitrage opportunity is a $\xi \in \Pi_{\mathbb{R}^d}$ with $c\xi = 0$ and $\xi^\top b > 0$ on a set of positive $\mathbb{P} \otimes G$ -measure. It follows that if X has continuous paths, immediate arbitrage opportunities do not exist if and only if b lies in the

range of c , i.e., if there exists a d -dimensional process ρ with $b = c\rho$; of course, if c is non-singular $\mathbb{P} \otimes G$ -almost everywhere, this always holds and $\rho = c^{-1}b$. It is easy to see that this “no immediate arbitrage opportunity” condition is equivalent to $dB_t \ll d[X, X]_t$. We refer the reader to [25], Appendix B of [27], and [11] for a more thorough discussion.

Remark 3.13 Let us write $X = A + M$ for the unique decomposition of a special semimartingale X into a predictable finite variation part A and a local martingale M , which we further assume is locally square-integrable. If $\langle M, M \rangle$ is the predictable compensator of $[M, M]$, Example 3.12 shows that in continuous-path models the condition for absence of immediate arbitrage is $dA_t \ll d\langle M, M \rangle_t$. Compare this with the more complicated way we have defined this notion in Definition 3.9. *Could the simple criterion $dA_t \ll d\langle M, M \rangle_t$ work in more general situations?* It is easy to see that $dA_t \ll d\langle M, M \rangle_t$ is necessary for the absence of immediate arbitrage opportunities; but it is not sufficient—it is too weak. Take, for example, X to be the standard Poisson process. In the non-constrained case, any positive portfolio is an immediate arbitrage opportunity. Nevertheless, $A_t = t$ and $M_t = X_t - t$ with $\langle M, M \rangle_t = t = A_t$, so that $dA_t \ll d\langle M, M \rangle_t$ holds trivially.

3.5 The growth-optimal portfolio and connection with the numéraire portfolio

We hinted in Remark 3.11 that if $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ is $\mathbb{P} \otimes G$ -null then one can find a process $\rho \in \Pi_{\mathcal{E}}$ such that $\text{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi_{\mathcal{E}}$. It is actually also important to have a way of computing this process ρ .

For a portfolio $\pi \in \Pi_{\mathcal{E}}$, its *growth rate* is defined as the drift rate of the log-wealth process $\log W^\pi$. One can use the stochastic exponential formula (1.1) and formally (since this will not always exist) compute the growth rate of W^π as

$$g(\pi) := \pi^\top b - \frac{1}{2} \pi^\top c \pi + \int [\log(1 + \pi^\top x) - \pi^\top x \mathbb{I}_{\{|x| \leq 1\}}] \nu(dx). \tag{3.4}$$

We describe (somewhat informally) the connection between the numéraire portfolio and the portfolio that maximizes in an (ω, t) -pointwise sense the growth rate over all portfolios in $\Pi_{\mathcal{E}}$ in the case of a deterministic triplet. Note that for the general semimartingale case this connection has already been observed in [18]. A $\rho \in \mathcal{C}$ maximizes the concave function g if and only if the derivative of g at the point ρ is negative in all directions $\pi - \rho$, $\pi \in \mathcal{C}$. This directional derivative is

$$\begin{aligned} (\nabla g)_\rho(\pi - \rho) &= (\pi - \rho)^\top b - (\pi - \rho)^\top c \rho \\ &\quad + \int \left[\frac{(\pi - \rho)^\top x}{1 + \rho^\top x} - (\pi - \rho)^\top x \mathbb{I}_{\{|x| \leq 1\}} \right] \nu(dx), \end{aligned}$$

which is exactly $\text{rel}(\pi \mid \rho)$. Of course, we do not know if we can differentiate under the integral appearing in (3.4). Even worse, we do not know a priori whether the integral is well-defined. Both its positive and negative parts could lead to infinite results. We now describe a class of Lévy measures for which the concave growth rate function $g(\cdot)$ of (3.4) is well-defined.

Definition 3.14 A Lévy measure ν for which $\int \log(1 + |x|) \mathbb{I}_{\{|x|>1\}} \nu(dx) < \infty$ will be said to *integrate the log*. Now consider any Lévy measure ν ; an *approximating sequence* is a sequence $(\nu_n)_{n \in \mathbb{N}}$ of Lévy measures that integrate the log with $\nu_n \sim \nu$, whose densities $f_n := d\nu_n/d\nu$ satisfy $0 < f_n \leq 1$, $f_n(x) = 1$ for $|x| \leq 1$, and $\uparrow\text{-lim}_{n \rightarrow \infty} f_n = \mathbb{I}$.

There are many ways to choose the sequence $(\nu_n)_{n \in \mathbb{N}}$, or, equivalently, the densities $(f_n)_{n \in \mathbb{N}}$; as a concrete example, take $f_n(x) = \mathbb{I}_{\{|x| \leq 1\}} + |x|^{-1/n} \mathbb{I}_{\{|x| > 1\}}$.

The integral in (3.4) is well defined and finite, when the Lévy measure ν integrates the log; and then ρ is the numéraire portfolio if and only if it maximizes $\mathfrak{g}(\cdot)$ pointwise. If ν fails to integrate the log, we shall consider a sequence of auxiliary problems as in Definition 3.14, then show that their solutions converge to the solution of the original problem.

3.6 The first main result

We are now ready to state the main result of this section, which solves Problem 3.3. We already discussed Condition (1) of Lemma 3.5 and its predictable characterization: *there exists a predictable process ρ with $\rho(\omega, t) \in \mathfrak{C}(\omega, t)$ such that $\text{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi_{\mathfrak{C}}$, if and only if $\{\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$ -measure (Remark 3.11).* If this holds, we construct such a process ρ ; the only thing that might keep ρ from being the numéraire portfolio is failure of X -integrability. To deal with this issue, define for a given predictable ρ

$$\psi^\rho := \nu[\rho^\top x > 1] + \left| \rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}}) \nu(dx) \right|.$$

Here is the statement of the main result; its proof is given in Sect. 6.

Theorem 3.15 *Consider a financial model described by a semimartingale returns process X and predictable closed convex constraints \mathfrak{C} .*

- (1.i) *If the predictable set $\{\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$ -measure then there exists a unique process $\rho \in \mathcal{P}(\mathbb{R}^d)$ with $\rho(\omega, t) \in \mathfrak{C} \cap \mathfrak{R}^\perp(\omega, t)$ for all $(\omega, t) \in \llbracket 0, T \rrbracket$ such that $\text{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi_{\mathfrak{C}}$.*
- (1.ii) *On the predictable set $\Lambda := \{\int \log(1 + |x|) \mathbb{I}_{\{|x|>1\}} \nu(dx) < \infty\}$, this process ρ is obtained as the unique solution of the concave optimization problem*

$$\rho = \arg \max_{\pi \in \mathfrak{C} \cap \mathfrak{R}^\perp} \mathfrak{g}(\pi).$$

In general, ρ can be obtained as the limit of solutions to corresponding problems (where one replaces ν by (ν_n) , an approximating sequence in the definition of \mathfrak{g}).

- (1.iii) *Furthermore, if the above constructed process $\rho \in \mathcal{P}(\mathbb{R}^d)$ is such that $(\psi^\rho \cdot G)_t < +\infty$ on $\llbracket 0, T \rrbracket$ then ρ is X -integrable and is the numéraire portfolio.*

2. Conversely to (1.i–1.ii–1.iii) above, if the numéraire portfolio ρ exists in $\Pi_{\mathcal{C}}$ then the predictable set $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$ -measure, and ρ satisfies $(\psi^\rho \cdot G)_t < +\infty$ on $\llbracket 0, T \rrbracket$, as well as $\text{rel}(\pi \mid \rho) \leq 0$ for all $\pi \in \Pi_{\mathcal{C}}$.

Let us pause to comment on the predictable characterization of X -integrability of ρ , which amounts to G -integrability of both processes

$$\psi_1^\rho := \nu[\rho^\top x > 1] \quad \text{and} \quad \psi_2^\rho := \rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}}) \nu(dx). \tag{3.5}$$

The integrability of ψ_1^ρ states that $\rho \cdot X$ cannot have an infinite number of large positive jumps on finite time-intervals; but this must hold, if $\rho \cdot X$ is to be well-defined. The second term ψ_2^ρ is exactly the drift rate of the part of $\rho \cdot X$ that remains when we subtract all large positive jumps (more than a unit in magnitude). This part has to be a special semimartingale, so its drift rate must be G -integrable, which is exactly the requirement $(|\psi_2^\rho| \cdot G)_t < \infty$ on $\llbracket 0, T \rrbracket$.

The requirement $\mathbb{P}[(\psi^\rho \cdot G)_t < +\infty \text{ on } \llbracket 0, T \rrbracket] = 1$ does *not* imply $(\psi^\rho \cdot G)_T < +\infty$ on $\{T = \infty\}$. The stronger requirement $(\psi^\rho \cdot G)_T < \infty$ means that ρ is X -integrable up to time T , in the terminology of Appendix 2. This, in turn, is equivalent to the fact that the numéraire portfolio exists *and* that $W_T^\rho < \infty$ (even on $\{T = \infty\}$). We shall return to this when we study arbitrage in the next section.

Remark 3.16 The conclusion of Theorem 3.15 can be stated succinctly as follows: the numéraire portfolio exists if and only if we have $\Psi(B, C, \eta) < \infty$ on $\llbracket 0, T \rrbracket$, for the *deterministic, increasing* functional $\Psi(B, C, \eta) := (\infty \mathbb{I}_{\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}} + \psi^\rho \mathbb{I}_{\{\mathcal{J} \cap \check{\mathcal{C}} = \emptyset\}}) \cdot G$ of the triplet of predictable characteristics (B, C, η) .

Example 3.17 Consider the unconstrained case $\mathcal{C} = \mathbb{R}^d$ for the continuous-path semimartingale case of Example 3.12. Since $(\nabla g)_\pi = b - c\pi = c\rho - c\pi$ is trivially zero for $\pi = \rho$, ρ will be the numéraire portfolio as long as $((\rho^\top c\rho) \cdot G)_t < \infty$ on $\llbracket 0, T \rrbracket$, or, in the case where c^{-1} exists, when $((b^\top c^{-1}b) \cdot G)_t < \infty$ on $\llbracket 0, T \rrbracket$.

3.7 Relative log-optimality

In this and the next subsection we give two optimality properties of the numéraire portfolio. Here we show that it is exactly the *log-optimal portfolio*, if the latter is defined in a relative sense.

Definition 3.18 A portfolio $\rho \in \Pi_{\mathcal{C}}$ will be called *relatively log-optimal*, if

$$\mathbb{E} \left[\limsup_{t \uparrow T} \left(\log \frac{W_t^\pi}{W_t^\rho} \right) \right] \leq 0 \quad \text{holds for every } \pi \in \Pi_{\mathcal{C}}.$$

Here the lim sup is clearly superfluous on $\{T < \infty\}$ but we include it to also cover the infinite time-horizon case. If ρ is relatively log-optimal, the lim sup is actually a finite limit; this is an easy consequence of the following result.

Proposition 3.19 *A numéraire portfolio exists if and only if a relatively log-optimal portfolio exists, in which case the two are the same.*

Proof Whenever we write $W_T^{\pi_1} / W_T^{\pi_2}$ for π_1 and π_2 in $\Pi_{\mathcal{E}}$, we tacitly imply that on $\{T = \infty\}$ the limit of this ratio exists, and take $W_T^{\pi_1} / W_T^{\pi_2}$ to be that limit.

- Suppose ρ is a numéraire portfolio. For any $\pi \in \Pi_{\mathcal{E}}$ we have $\mathbb{E}[W_T^{\pi} / W_T^{\rho}] \leq 1$, and Jensen’s inequality gives $\mathbb{E}[\log(W_T^{\pi} / W_T^{\rho})] \leq 0$, so ρ is also relatively log-optimal.
- Let us now assume that a relative log-optimal portfolio $\hat{\rho}$ exists—we shall show that the numéraire portfolio exists and is equal to $\hat{\rho}$. Without loss of generality, assume that $\hat{\rho}(\omega, t)$ lies in $\mathfrak{N}(\omega, t)$ for $\mathbb{P} \otimes G$ -almost every $(\omega, t) \in \llbracket 0, T \rrbracket$ —otherwise, we project $\hat{\rho}(\omega, t)$ on $\mathfrak{N}(\omega, t)$ and observe that the projected portfolio generates the same wealth as the original.

First, we observe that $\{\mathfrak{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ must have zero $\mathbb{P} \otimes G$ -measure. To see why, suppose the contrary. Then, by Proposition 3.10, we could select a portfolio $\xi \in \Pi_{\mathcal{E}}$ that leads to unbounded increasing profit. According to Remark 3.11, we would have $\hat{\rho} + \xi \in \Pi_{\mathcal{E}}$ and $\text{rel}(\hat{\rho} \mid \hat{\rho} + \xi) \leq 0$, with strict inequality on a predictable set of positive $\mathbb{P} \otimes G$ -measure; this would mean that the process $W^{\hat{\rho}} / W^{\hat{\rho} + \xi}$ is a non-constant positive supermartingale, and Jensen’s inequality again would give $\mathbb{E}[\log(W_T^{\hat{\rho}} / W_T^{\hat{\rho} + \xi})] < 0$, contradicting the relative log-optimality of $\hat{\rho}$.

Continuing, since $\{\mathfrak{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$ -measure, we can construct the predictable process ρ which is the candidate in Theorem 3.15 (1) for being the numéraire portfolio. We only need to show that the predictable set $\{\hat{\rho} \neq \rho\}$ has zero $\mathbb{P} \otimes G$ -measure. By way of contradiction, suppose that $A_n := \{\hat{\rho} \neq \rho, |\rho| \leq n\}$ has non-zero $\mathbb{P} \otimes G$ -measure for some $n \in \mathbb{N}$. We then define $\pi := \hat{\rho} \mathbb{I}_{\llbracket 0, T \rrbracket \setminus A_n} + \rho \mathbb{I}_{A_n} \in \Pi_{\mathcal{E}}$ —since $\text{rel}(\hat{\rho} \mid \pi) = \text{rel}(\hat{\rho} \mid \rho) \mathbb{I}_{A_n} \leq 0$ with strict inequality on A_n , the same argument as in the end of the preceding paragraph shows that $\hat{\rho}$ cannot be the relatively log-optimal portfolio. We conclude that $\{\hat{\rho} \neq \rho\} = \bigcup_{n \in \mathbb{N}} A_n$ has zero $\mathbb{P} \otimes G$ -measure, and thus $\rho = \hat{\rho}$ is the numéraire portfolio. □

We remark that if the log-utility optimization problem has a finite value and the condition NFLVR of Delbaen and Schachemayer [10] holds (see also Definition 4.1 below), the result of the last proposition is well-known—see, for example, [30]. Christensen and Larsen [8] start by adopting the above “relative” definition as log-optimality (or, as they call it, “growth optimality”) and eventually show that the growth-optimal is equal to the numéraire portfolio.

Example 3.20 Take a one-stock market model with $S_t = \exp(\beta_{T \wedge t})$, where β is a standard, one-dimensional Brownian motion and T an a.s. finite stopping time with $\mathbb{E}[\beta_T^-] < +\infty$ and $\mathbb{E}[\beta_T^+] = +\infty$. Then $\mathbb{E}[\log S_T] = +\infty$ and the classical log-utility optimization problem is not well-posed (one can find a multitude of portfolios with infinite expected log-utility). In this case, Example 3.12 shows that $\rho = 1/2$ is both the numéraire and the relative log-optimal portfolio.

3.8 An asymptotic optimality property

In this subsection we deal with a purely infinite time-horizon case $T \equiv \infty$ and describe an “asymptotic growth optimality” property of the numéraire portfolio ρ , which we assume to exist. Note that for any portfolio $\pi \in \Pi_{\mathcal{E}}$ the process W^π / W^ρ is a positive supermartingale and, therefore, $\lim_{t \rightarrow \infty} (W_t^\pi / W_t^\rho)$ exists in $[0, +\infty)$. Consequently, for any increasing process H with $H_\infty = +\infty$ (H does not even have to be adapted), we have $\limsup_{t \rightarrow \infty} ((H_t)^{-1} \log(W_t^\pi / W_t^\rho)) \leq 0$. A closely-related version of “asymptotic growth optimality” was first observed and proved in Algoet and Cover [1] for the discrete-time case; see also [27], Sect. 3.10, and [18] for a discussion of this asymptotic optimality in the continuous-path and the general semimartingale case, respectively. In the above-mentioned works, the authors prove that $\limsup_{t \rightarrow \infty} (t^{-1} \log W_t^\pi) \leq \limsup_{t \rightarrow \infty} (t^{-1} \log W_t^\rho) \leq 0$, which is certainly a weaker statement than what we mention (interestingly, the proof used is more involved, using a “Borel–Cantelli”-type argument).

Our next result, Proposition 3.21, separates the cases when $\lim_{t \rightarrow \infty} (W_t^\pi / W_t^\rho)$ is $(0, \infty)$ -valued and when it is zero, and describes this dichotomy in terms of predictable characteristics. In the case of convergence to zero, it quantifies how fast this convergence takes place. Its proof is given in Sect. 7.

Proposition 3.21 *Assume that the numéraire portfolio ρ exists on $\llbracket 0, \infty \rrbracket$. For any other $\pi \in \Pi_{\mathcal{E}}$, define the positive, predictable process*

$$h^\pi := -\text{rel}(\pi \mid \rho) + \frac{1}{2}(\pi - \rho)^\top c(\pi - \rho) + \int q_a \left(\frac{1 + \pi^\top x}{1 + \rho^\top x} \right) \nu(dx),$$

and the increasing, predictable process $H^\pi := h^\pi \cdot G$. Here we use the positive, convex function $q_a(y) := [-\log a + (1 - a^{-1})y] \mathbb{I}_{[0, a)}(y) + [y - 1 - \log y] \mathbb{I}_{[a, +\infty)}(y)$ for some $a \in (0, 1)$. Then, on $\{H_\infty^\pi < +\infty\}$, $\lim_{t \rightarrow \infty} (W_t^\pi / W_t^\rho) \in (0, +\infty)$, while

$$\text{on } \{H_\infty^\pi = +\infty\}, \quad \limsup_{t \rightarrow \infty} \left(\frac{1}{H_t^\pi} \log \frac{W_t^\pi}{W_t^\rho} \right) \leq -1.$$

Remark 3.22 Some comments are in order. We begin with the “strange-looking” function $q_a(\cdot)$, that depends also on the (cut-off point) parameter $a \in (0, 1)$. Ideally we should like to define $q_0(y) = y - 1 - \log y$ for all $y > 0$, since then the predictable increasing process H^π would be exactly the negative of the drift of the semimartingale $\log(W^\pi / W^\rho)$. Unfortunately, a problem arises when the positive predictable process $\int q_0(\frac{1 + \pi^\top x}{1 + \rho^\top x}) \nu(dx)$ fails to be G -integrable, which is equivalent to saying that $\log(W^\pi / W^\rho)$ is not a special semimartingale; the problem comes from the fact that $q_0(y)$ explodes to $+\infty$, as $y \downarrow 0$. For this reason, we define $q_a(\cdot)$ to be equal to $q_0(\cdot)$ on $[a, \infty)$, linear on $[0, a)$, and continuously differentiable at the “glueing” point a . The functions $q_a(\cdot)$ are all finite-valued at $y = 0$ and satisfy $q_a(\cdot) \uparrow q_0(\cdot)$, as $a \downarrow 0$.

Let us now study h^π and H^π . Observe that h^π is predictably convex in π , namely, if π_1 and π_2 are two portfolios and λ is a $[0, 1]$ -valued predictable process then

$h^{\lambda\pi_1+(1-\lambda)\pi_2} \leq \lambda h^{\pi_1} + (1-\lambda)h^{\pi_2}$. This, together with the fact that $h^\pi = 0$ if and only if $\pi - \rho$ is a null investment, casts h^π as a *measure of instantaneous deviation of π from ρ* ; by the same token, H_∞^π can be seen as the total (cumulative) deviation of π from ρ . With this in mind, Proposition 3.21 says that, if an investment deviates a lot from the numéraire portfolio ρ (i.e., if $H_\infty^\pi = +\infty$), its long-term performance will lag considerably behind that of ρ . Only if an investment tracks very closely the numéraire portfolio over $[0, \infty)$ (i.e., if $H_\infty^\pi < +\infty$) will the two wealth processes have comparable growth. Letting $a \downarrow 0$ in the definition of H^π we get equivalent measures of distance of a portfolio π from the numéraire portfolio because $\{H_\infty^\pi = +\infty\}$ does not depend on the choice of a ; nevertheless, we get ever sharper results, since h^π is increasing for decreasing $a \in (0, 1)$.

4 Unbounded profits with bounded risks, supermartingale deflators, and the numéraire portfolio

In this section we proceed to investigate how the existence or non-existence of the numéraire portfolio relates to some concept of “free lunch” in the financial market. We shall eventually prove a version of the Fundamental Theorem of Asset Pricing; this is our second main result, Theorem 4.12.

4.1 Arbitrage-type definitions

We first recall two widely known no-free-lunch conditions for financial markets (NA and the stronger NFLVR), together with yet another notion which is exactly what one needs to bridge the gap between the previous two, and will actually be the most important for our discussion.

Definition 4.1 For the following definitions we consider our financial model with constraints \mathcal{C} on portfolios. When we write W_T^π for some $\pi \in \Pi_{\mathcal{C}}$ we assume tacitly that $\lim_{t \rightarrow \infty} W_t^\pi$ exists on $\{T = \infty\}$, and set W_T^π equal to that limit.

- A portfolio $\pi \in \Pi_{\mathcal{C}}$ is said to generate an *arbitrage opportunity*, if $\mathbb{P}[W_T^\pi \geq 1] = 1$ and $\mathbb{P}[W_T^\pi > 1] > 0$. If no such portfolio exists we say that the \mathcal{C} -constrained market satisfies the *No Arbitrage* condition, which we denote by $\text{NA}_{\mathcal{C}}$.
- A sequence $(\pi_n)_{n \in \mathbb{N}}$ of portfolios in $\Pi_{\mathcal{C}}$ is said to generate an *unbounded profit with bounded risk* (UPBR), if the collection of positive random variables $(W_T^{\pi_n})_{n \in \mathbb{N}}$ is unbounded in probability, i.e., if $\downarrow\text{-lim}_{m \rightarrow \infty} \sup_{n \in \mathbb{N}} \mathbb{P}[W_T^{\pi_n} > m] > 0$. If no such sequence exists, we say that the constrained market satisfies the *no unbounded profit with bounded risk* (NUPBR $_{\mathcal{C}}$) condition.
- A sequence $(\pi_n)_{n \in \mathbb{N}}$ of portfolios in $\Pi_{\mathcal{C}}$ is said to be a *free lunch with vanishing risk* (FLVR), if there exist an $\epsilon > 0$ and an increasing sequence $(\delta_n)_{n \in \mathbb{N}}$ with $0 \leq \delta_n \uparrow 1$, such that $\mathbb{P}[W_T^{\pi_n} > \delta_n] = 1$ as well as $\mathbb{P}[W_T^{\pi_n} > 1 + \epsilon] \geq \epsilon$. If no such sequence exists, we say that the market satisfies the *no free lunch with vanishing risk* (NFLVR $_{\mathcal{C}}$) condition.

The NFLVR condition was introduced by Delbaen and Schachermayer [10] in a slightly different form. With the above definition of FLVR and the convexity Lemma A.1.1 from [10], we can further assume that there exists a $[1, +\infty]$ -valued random variable f with $\mathbb{P}[f > 1] > 0$ such that $\mathbb{P}\text{-}\lim_{n \rightarrow \infty} W_T^{\pi_n} = f$; this brings us back to the usual definition in [10].

If an UPBR exists, one can find a sequence of wealth processes, each starting with less and less capital (converging to zero) and such that the terminal wealths are unbounded with a fixed probability. Thus, UPBR can be translated as “the *possibility* of making (a considerable) amount out of almost nothing”; it should be contrasted with the classical notion of arbitrage, which can be translated as “the *certainty* of making something more out of something”.

Observe that $\text{NUPBR}_{\mathcal{C}}$ can be alternatively stated by using portfolios with *bounded support*, so the requirement of a limit at infinity for the wealth processes on $\{T = \infty\}$ is automatically satisfied. This is relevant because, as we shall see, when $\text{NUPBR}_{\mathcal{C}}$ holds, every wealth process W^π has a limit on $\{T = \infty\}$ and is a semimartingale up to T in the terminology of Appendix 2.

None of the two conditions $\text{NA}_{\mathcal{C}}$ and $\text{NUPBR}_{\mathcal{C}}$ implies the other, and they are not mutually exclusive. It is easy to see that they are both weaker than $\text{NFLVR}_{\mathcal{C}}$, and that, in fact, we have the following result which gives the exact relationships among these notions under cone constraints. Its proof can be found in [10] for the unconstrained case; we include it here for completeness.

Proposition 4.2 *Suppose that \mathcal{C} enforces predictable closed convex cone constraints. Then $\text{NFLVR}_{\mathcal{C}}$ holds if and only if both $\text{NA}_{\mathcal{C}}$ and $\text{NUPBR}_{\mathcal{C}}$ hold.*

Proof It is obvious that if either $\text{NA}_{\mathcal{C}}$ or $\text{NUPBR}_{\mathcal{C}}$ fail then $\text{NFLVR}_{\mathcal{C}}$ fails too. Conversely, suppose that $\text{NFLVR}_{\mathcal{C}}$ fails. If $\text{NA}_{\mathcal{C}}$ fails there is nothing more to say, so suppose that $\text{NA}_{\mathcal{C}}$ holds and let $(\pi_n)_{n \in \mathbb{N}}$ generate a free lunch with vanishing risk. Under $\text{NA}_{\mathcal{C}}$, the assumption $\mathbb{P}[W_T^{\pi_n} > \delta_n] = 1$ results in the stronger $\mathbb{P}[W_t^{\pi_n} > \delta_n \text{ for all } t \in [0, T]] = 1$. Construct a new sequence of wealth processes $(W^{\xi_n})_{n \in \mathbb{N}}$ by requiring $W^{\xi_n} = 1 + (1 - \delta_n)^{-1}(W^{\pi_n} - 1)$, check that $W^{\xi_n} > 0$, and then that $\xi_n \in \Pi_{\mathcal{C}}$ (here it is essential that \mathcal{C} be a cone). Furthermore, $\mathbb{P}[W_T^{\pi_n} \geq 1 + \epsilon] \geq \epsilon$ becomes $\mathbb{P}[W_T^{\xi_n} > 1 + (1 - \delta_n)^{-1}\epsilon] \geq \epsilon$; thus, $(\xi_n)_{n \in \mathbb{N}}$ generates an unbounded profit with bounded risk and $\text{NUPBR}_{\mathcal{C}}$ fails. □

4.2 Fundamental theorem of asset pricing (FTAP)

The $\text{NFLVR}_{\mathcal{C}}$ condition has proven very fruitful in contexts where we can change the original measure \mathbb{P} to some other equivalent probability \mathbb{Q} , under which the wealth processes have some kind of (super)martingale property.

Definition 4.3 Consider a financial market model described by a semimartingale discounted stock price process S and predictable closed convex constraints \mathcal{C} on portfolios. A probability \mathbb{Q} will be called a *equivalent \mathcal{C} -supermartingale measure* ($\text{ESMM}_{\mathcal{C}}$, for short), if $\mathbb{Q} \sim \mathbb{P}$ on \mathcal{F}_T , and if W^π is a \mathbb{Q} -supermartingale for every $\pi \in \Pi_{\mathcal{C}}$. The class of $\text{ESMM}_{\mathcal{C}}$ is denoted by $\mathfrak{M}_{\mathcal{C}}$.

Similarly, define a *equivalent \mathcal{C} -local martingale measure* (ELMM $_{\mathcal{C}}$, for short) \mathbb{Q} by requiring $\mathbb{Q} \sim \mathbb{P}$ on \mathcal{F}_T and that W^π be a \mathbb{Q} -local martingale for every $\pi \in \Pi_{\mathcal{C}}$.

In Definition 4.3 we might as well assume that \mathcal{C} are cone constraints; because, if ESMM $_{\mathcal{C}}$ holds the same holds for the market under constraints $\overline{\text{con}}(\mathcal{C})$, the closure of the cone generated by \mathcal{C} .

The following result is one of the best-known in mathematical finance; we present its “cone-constrained” version.

Theorem 4.4 (FTAP) *For a financial market model with stock-price process S and predictable closed convex cone constraints \mathcal{C} , NFLVR $_{\mathcal{C}}$ is equivalent to $\mathfrak{M}_{\mathcal{C}} \neq \emptyset$.*

Because we are working under constraints, we cannot hope, in general, for anything better than an equivalent *supermartingale* measure in the statement of Theorem 4.4. One can see this easily in the case where X is a single-jump process which jumps at a stopping time τ with $\Delta X_\tau \in (-1, 0)$ and we are constrained in the cone of positive strategies. Under any measure $\mathbb{Q} \sim \mathbb{P}$, the process $S = \mathcal{E}(X) = W^1$, an admissible wealth process, will be non-increasing and not identically zero; this prevents it from being even a local martingale.

The implication $\mathfrak{M}_{\mathcal{C}} \neq \emptyset \Rightarrow \text{NFLVR}_{\mathcal{C}}$ is easy; the reverse is considerably harder for the general semimartingale model. Several papers are devoted to proving some version of Theorem 4.4; in the generality assumed here, a proof appears in [22], although all the crucial work was done by Delbaen and Schachermayer [10] and the theorem is certainly due to them. Theorem 4.4 can be derived from Kabanov’s statement, since the class of wealth processes $(W^\pi)_{\pi \in \Pi_{\mathcal{C}}}$ is convex and closed in the semimartingale (also called “Émery”) topology. A careful inspection in Mémin’s work [34] of the proof that the set of all stochastic integrals with respect to the d -dimensional semimartingale X is closed under this topology, shows that one can pick the limiting semimartingale from a convergent sequence $(W^{\pi_n})_{n \in \mathbb{N}}$, with $\pi_n \in \Pi_{\mathcal{C}}$ for all $n \in \mathbb{N}$, to be of the form W^π for some $\pi \in \Pi_{\mathcal{C}}$.

4.3 Beyond the fundamental theorem of asset pricing

Let us study some more the assumptions and the statement of Theorem 4.4. We shall be concerned with three questions, which will turn out to have the same answer; this answer will be linked with the NUPBR $_{\mathcal{C}}$ condition and—as we shall see in Theorem 4.12—with the existence of the numéraire portfolio.

4.3.1 Convex but non-conic constraints

In the statement of Theorem 4.4 it is *crucial* that the constraint be a *cone*—the result fails without the “cone” assumption. Of course, $\mathfrak{M}_{\mathcal{C}} \neq \emptyset \Rightarrow \text{NFLVR}_{\mathcal{C}}$ still holds, but the reverse does not, as shown in the example below (a raw version of a similar example from [29]).

Example 4.5 Consider two stocks with discounted prices S^1 and S^2 in a simple one-period, discrete-time model. We have $S_0^1 = S_0^2 = 1$, while $S_1^1 = 1 + e$ and $S_1^2 = f$.

Here e and f are two independent, exponentially distributed random variables. The class of portfolios is easily identified with all $(p, q) \in \mathcal{C}_0 = \mathbb{R}_+ \times [0, 1]$. Since $X_1^1 = S_1^1 - S_0^1 = e > 0$, \mathbb{P} -a.s., we have that NA fails for this (unconstrained) market. In other words, for the non-constrained case there can be no ESMM.

Consider now the non-random constraint set $\mathcal{C} = \{(p, q) \in \mathcal{C}_0 \mid p^2 \leq q\}$. Observe that $\overline{\text{cone}}(\mathcal{C}) = \mathbb{R}_+ \times \mathbb{R}$ and thus no ESMM $_{\mathcal{C}}$ exists; otherwise, an ESMM would exist already for the unconstrained case. We shall, nevertheless, show in the following paragraph that NFLVR $_{\mathcal{C}}$ holds for this constrained market.

For a sequence of portfolios $\pi_n \equiv (p_n, q_n)_{n \in \mathbb{N}} \in \mathcal{C}$, the wealth on day one will be $W_1^{\pi_n} = 1 - q_n + q_n f + p_n e$; obviously, $\mathbb{P}[W_1^{\pi_n} \geq 1 - q_n] = 1$, since $1 - q_n$ is the essential infimum of $W_1^{\pi_n}$. It then turns out that in order for $(\pi_n)_{n \in \mathbb{N}}$ to generate a FLVR we must require $q_n \downarrow 0$ and $\mathbb{P}[W_1^{\pi_n} > 1 + \epsilon] > \epsilon$ for some $\epsilon > 0$. Observe that we *must* have $q_n > 0$, otherwise, $p_n = 0$ as well (because of the constraints), and then $W_1^{\pi_n} = 1$. Now, because of the constraints, again we have $|p_n| \leq \sqrt{q_n}$; since $\mathbb{P}[e > 0] = 1$, the sequence of strategies $\xi_n := (\sqrt{q_n}, q_n)$ will generate a sequence of wealth processes $(W^{\xi_n})_{n \in \mathbb{N}}$ that will dominate $(W^{\pi_n})_{n \in \mathbb{N}}$, i.e., $\mathbb{P}[W_1^{\xi_n} \geq W_1^{\pi_n}] = 1$; this will, of course, mean that $(W^{\xi_n})_{n \in \mathbb{N}}$ is also a FLVR. We should then have $\mathbb{P}[1 - q_n + \sqrt{q_n}e + q_n f > 1 + \epsilon] > \epsilon$; using $q_n > 0$ and some algebra we get $\mathbb{P}[e > \sqrt{q_n}(1 - f) + \epsilon/\sqrt{q_n}] > \epsilon$. Since $(q_n)_{n \in \mathbb{N}}$ goes to zero, this would imply that $\mathbb{P}[e > M] \geq \epsilon$ for all $M > 0$, which is clearly ridiculous. We conclude that NFLVR $_{\mathcal{C}}$ holds, although as we have seen $\mathfrak{M}_{\mathcal{C}} = \emptyset$.

What can we say in the case of convex—but non necessarily conic—constraints? It will turn out that for the equivalent of the FTAP, the assumptions from *both* the economic *and* the mathematical side should be relaxed. The relevant economic notion will be NUPBR $_{\mathcal{C}}$ and the mathematical one will be the concept of supermartingale deflators—more on this in Sects. 4.4 and 4.5.

4.3.2 Describing free lunches in terms of predictable characteristics

The explanation why “free lunches” are considered economically unsound stems from the following reasoning: if they exist in a market, many agents will try to take advantage of them; then, usual supply-and-demand arguments will imply that some correction on the prices of the assets will occur and remove these kinds of opportunities. This is a very reasonable line of thought, *provided that one can discover the free lunches that are present*. But is it true that, given a specific model, one is in a position to decide whether free lunches exist or not? In other words, mere knowledge of the *existence* of a free lunch may not be enough to carry the previous economic argument—one should be able to *construct* a free lunch. This goes somewhat hand in hand with the fact that the FTAP is a *pure existence* result, in the sense that it provides knowledge that *some* equivalent (super)martingale measure exists; in some cases one might be able to spot it, in other cases might not.

A natural question arises: *when free lunches exist, is there a way to construct them from the predictable characteristics of the model?* Here is an answer: if NUPBR $_{\mathcal{C}}$ fails then an UPBR can be constructed using the triplet (B, C, η) . The detailed statement will be given in Sect. 4.6, but let us say here that the *deterministic* positive

functional Ψ of Remark 3.16 is such that on the event $\{\Psi_T(B, C, \eta) = \infty\}$ $\text{NUPBR}_{\mathcal{E}}$ fails (and then we can construct free lunches using the predictable characteristics), while on $\{\Psi_T(B, C, \eta) < \infty\}$ $\text{NUPBR}_{\mathcal{E}}$ holds. As a result, we see that $\text{NUPBR}_{\mathcal{E}}$ is somehow a *pathwise* notion.

What we described in the last paragraph for the $\text{NUPBR}_{\mathcal{E}}$ condition does not apply to the $\text{NA}_{\mathcal{E}}$ condition, as we demonstrate in Example 4.7.

Example 4.6 (Arbitrage for the three-dimensional Bessel Process) Consider a one-stock market on the finite time horizon $[0, 1]$, with $S_0 = 1$ and S satisfying the stochastic differential equation $dS_t = (1/S_t) dt + d\beta_t$. Here β is a standard, one-dimensional Brownian motion, so S is the *three-dimensional Bessel process*. Writing $dS_t/S_t = (1/S_t^2) dt + (1/S_t) d\beta_t =: dX_t$ and using Example 3.17, the numéraire portfolio for the unconstrained case exists and is $\rho = 1$.

This market admits arbitrage. To wit, with the notation

$$\Phi(x) = \int_{-\infty}^x \frac{e^{-u^2/2}}{\sqrt{2\pi}} du, \quad F(t, x) = \frac{\Phi(x/\sqrt{1-t})}{\Phi(1)}, \quad \text{for } x \in \mathbb{R} \text{ and } 0 < t < 1,$$

consider the process $W_t = F(t, S_t)$. Obviously $W_0 = 1, W > 0$ and

$$dW_t = \frac{\partial F}{\partial x}(t, S_t) dS_t, \quad \text{and thus} \quad \frac{dW_t}{W_t} = \left[\frac{1}{F(t, S_t)} \frac{\partial F}{\partial x}(t, S_t) \right] dS_t,$$

by Itô's formula. We conclude that $W = W^\pi$ for $\pi_t := (\partial \log F / \partial x)(t, S_t)$, and that $W_1^\pi = 1/\Phi(1) > 1$, i.e., there exists arbitrage in the market.

We remark that there is also an indirect way to show that arbitrage exists using the FTAP, proposed by Delbaen and Schachermayer [13]; there, one has to *further* assume that the filtration \mathbf{F} is the one generated by S (equivalently, by β).

This is one of the rare occasions when one can compute the arbitrage portfolio concretely. We were successful in this because of the very special structure of the three-dimensional Bessel process; every model has to be attacked in a different way, and there is no general theory that will spot the arbitrage. Nevertheless, we refer the reader to Fernholz, Karatzas, and Kardaras [16] and Fernholz and Karatzas [15] for many examples of arbitrage relative to the *market portfolio* (whose wealth process follows exactly the index $\sum_{i=1}^d S^i$ in proportion to the initial investment). This is done under conditions on the market structure that are easy to check, and are descriptive—as opposed to normative, such as ELMM.

We now show that there *cannot* exist a deterministic positive functional Ψ that takes as its arguments triplets of predictable characteristics such that NA holds, whenever $\mathbb{P}[\Psi_T(B, C, \eta) < \infty] = 1$. Actually, we shall construct in the next paragraph two stock-price processes on the *same* stochastic basis and with the *same* predictable characteristics, and such that NA fails with respect to the one but holds with respect to the other.

Example 4.7 (No predictable characterization of arbitrage) Assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is rich enough to accommodate two independent standard one-dimensional Brownian

motions β and γ ; the filtration will be the (usual augmentation of the) one generated by the pair (β, γ) . We work in the time-horizon $[0, 1]$. Let R be the three-dimensional Bessel process with $R_0 = 1$ and $dR_t = (1/R_t) dt + d\beta_t$. As R is adapted to the filtration generated by β , it is independent of γ . Start with the market described by the stock-price $S = R$; the triplet of predictable characteristics (B, C, η) consists of $B_t = C_t = \int_0^t (1/R_u)^2 du$ and $\eta = 0$. According to Example 4.6, NA fails for this market.

With the same process R , define now a new stock \widehat{S} following the dynamics $d\widehat{S}_t/\widehat{S}_t = (1/R_t)^2 dt + (1/R_t)d\gamma_t$ with $\widehat{S}_0 = 1$. The new dynamics involve γ , so \widehat{S} is *not* a three-dimensional Bessel process; nevertheless, it has exactly the same triplet of predictable characteristics as S . But now NA holds for the market that consists of the stock \widehat{S} . We can actually construct an ELMM, since the independence of R and γ implies that the exponential local martingale $Z := \mathcal{E}(-(1/R) \cdot \gamma)$ is a *true* martingale; Lemma 4.8 below will show this. We can then define $\mathbb{Q} \sim \mathbb{P}$ via $d\mathbb{Q}/d\mathbb{P} = Z_1$, and Girsanov’s theorem will imply that \widehat{S} is the stochastic exponential of a Brownian motion under \mathbb{Q} —thus, a true martingale.

Lemma 4.8 *On a stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t \in \mathbb{R}_+}, \mathbb{P})$ let β be a standard one-dimensional \mathbf{F} -Brownian motion and α a predictable process, independent of β , that satisfies $\int_0^t |\alpha_u|^2 du < \infty$, \mathbb{P} -a.s. Then the exponential local martingale $Z = \mathcal{E}(\alpha \cdot \beta)$ satisfies $\mathbb{E}[Z_t] = 1$, i.e., is a true martingale on $[0, t]$.*

Proof We begin by enlarging the filtration to \mathbf{G} with $\mathcal{G}_t := \mathcal{F}_t \vee \sigma(\alpha_t; t \in \mathbb{R}_+)$, i.e., we throw the *whole* history of α up to the end of time in \mathbf{F} . Since α and β are independent, it is easy to see that β is a \mathbf{G} -Brownian motion. Of course, α is a \mathbf{G} -predictable process and thus the stochastic integral $\alpha \cdot \beta$ is the same seen under \mathbf{F} or \mathbf{G} . Then, with $A_n := \{n - 1 \leq \int_0^t |\alpha_u|^2 du < n\} \in \mathcal{G}_0$ and in view of $\mathbb{E}[Z_t | A_n] = 1$ (since on A_n the quadratic variation of $\alpha \cdot \beta$ is bounded by n), we have $\mathbb{E}[Z_t] = \mathbb{E}[\mathbb{E}[Z_t | \mathcal{G}_0]] = \sum_{n=1}^\infty \mathbb{E}[Z_t | A_n] \mathbb{P}[A_n] = 1$. □

4.3.3 Connection with utility maximization

A central problem of mathematical finance is the maximization of *expected utility from terminal wealth* of an economic agent who can invest in the market. The agent’s preferences are described by a *utility function*: namely, a *concave* and *strictly increasing* function $U : (0, \infty) \mapsto \mathbb{R}$. We also define $U(0) \equiv U(0+)$ by continuity. Starting with initial capital $w > 0$, the objective of the investor is to find a portfolio $\hat{\rho} \equiv \hat{\rho}(w) \in \Pi_{\mathcal{E}}$ such that

$$\mathbb{E}[U(wW_T^{\hat{\rho}})] = \sup_{\pi \in \Pi_{\mathcal{E}}} \mathbb{E}[U(wW_T^\pi)] =: u(w). \tag{4.1}$$

Probably the most important example is the logarithmic utility function $U(w) = \log w$. Due to its special structure, when the optimal portfolio exists it does not depend on the initial capital, nor on the given time-horizon T (“myopia”). We saw in Sect. 3.7 that, under a suitable reformulation of log-optimality, the two notions of log-optimal and numéraire portfolio are equivalent.

We consider here utility maximization from terminal wealth that is constrained to be positive (in other words, $U(w) = -\infty$ for $w < 0$). This problem has a long history; it has been solved in a very satisfactory manner for general semimartingale models using previously-developed ideas of martingale duality by Kramkov and Schachermayer [30, 31], where we send the reader for further details.

A common assumption in this context is that the class of equivalent local martingale measures is non-empty, i.e., that NFLVR holds. (Interestingly, in [26] this assumption is *not* made.) The three-dimensional Bessel process of Example 4.6 shows that this is not necessary; indeed, since the numéraire portfolio $\rho = 1$ exists and $\mathbb{E}[\log S_1] < \infty$, Proposition 3.19 shows that ρ is the solution to the log-utility optimization problem. Nevertheless, we have seen that NFLVR fails for this market. To wit: an investor with log-utility will optimally choose to hold the stock and, *even though arbitrage opportunities exist in the market, the investor’s optimal choice is clearly not an arbitrage.*

In the mathematical theory of economics, the equivalence of no free lunches, equivalent martingale measures, and the existence of optimal investments for utility-based preferences is something of a “folklore theorem”. Theorem 4.4 deals with the equivalence of the first two of these conditions, but the three-dimensional Bessel process example shows that this does not completely cover minimal conditions for utility maximization; in that example, although NA fails, the numéraire and log-optimal portfolios do exist. In Theorem 4.12 we shall see that *existence of the numéraire portfolio is equivalent to the NUPBR condition* and (in Sect. 4.7) that NUPBR is actually the *minimal* “no free lunch”-type notion needed to ensure the existence of solution to *any* utility maximization problem. In a loose sense (to become precise there) the problem of maximizing expected utility from terminal wealth is solvable for a rather large class of utility functions *if and only if* the special case of the logarithmic utility problem has a solution—which is exactly when NUPBR holds. Accordingly, the existence of an equivalent (local) martingale measure will have to be substituted by the weaker requirement, the existence of a *supermartingale deflator*, which is the subject of the next subsection.

4.4 Supermartingale deflators

In the spirit of Theorem 4.4, we should now like to find a mathematical condition equivalent to NUPBR. The next concept, closely related to that of equivalent supermartingale measures but weaker, will be exactly what we shall need.

Definition 4.9 The class of *equivalent supermartingale deflators* is defined as

$$\mathcal{D}_{\mathcal{E}} := \{D \geq 0 \mid D_0 = 1, D_T > 0, \text{ and } DW^\pi \text{ is supermartingale } \forall \pi \in \Pi_{\mathcal{E}}\}.$$

If there exists an element $D^* \in \mathcal{D}_{\mathcal{E}}$ of the form $D^* \equiv 1/W^\rho$ for some $\rho \in \Pi_{\mathcal{E}}$, we call D^* a *tradeable supermartingale deflator*.

If a tradeable supermartingale deflator $D^* \equiv 1/W^\rho$ exists then the relative wealth process W^π / W^ρ is a supermartingale for all $\pi \in \Pi_{\mathcal{E}}$, i.e., ρ is the numéraire portfolio. Thus, a tradeable supermartingale deflator exists if and only if a numéraire portfolio ρ exists and $W_T^\rho < \infty$, \mathbb{P} -a.s.; and then it is unique.

An equivalent supermartingale measure \mathbb{Q} generates an equivalent supermartingale deflator through the positive martingale $D_t = (d\mathbb{Q}/d\mathbb{P})|_{\mathcal{F}_t}$. Then we have $\mathfrak{M}_{\mathcal{E}} \subseteq \mathfrak{D}_{\mathcal{E}}$ (for the class $\mathfrak{M}_{\mathcal{E}}$ of equivalent \mathcal{E} -supermartingale measures of Definition 4.3); thus, $\mathfrak{M}_{\mathcal{E}} \neq \emptyset \Rightarrow \mathfrak{D}_{\mathcal{E}} \neq \emptyset$. In general, the elements of $\mathfrak{D}_{\mathcal{E}}$ are just supermartingales, not martingales, and the inclusion $\mathfrak{M}_{\mathcal{E}} \subseteq \mathfrak{D}_{\mathcal{E}}$ is strict; more importantly, the implication $\mathfrak{D}_{\mathcal{E}} \neq \emptyset \Rightarrow \mathfrak{M}_{\mathcal{E}} \neq \emptyset$ does *not* hold, as we now show.

Example 4.10 Consider the three-dimensional Bessel process of Example 4.6 on the finite time-horizon $[0, 1]$. Since $\rho = 1$ is the numéraire portfolio, $D^* = 1/S$ is a tradeable supermartingale deflator, so $\mathfrak{D}_{\mathcal{E}} \neq \emptyset$. As we have already seen, NA fails; thus, we must have $\mathfrak{M}_{\mathcal{E}} = \emptyset$.

The set $\mathfrak{D}_{\mathcal{E}}$ of equivalent supermartingale deflators appears as the range of optimization in the “dual” of the utility maximization problem (4.1) in [30]. It has appeared before in some generalization of Kramkov’s optional decomposition theorem by Stricker and Yan [38], as well as in [36] under the name “martingale densities” (in both of these works, \mathfrak{D} consisted of positive local martingales).

As we shall see soon, *it is the condition $\mathfrak{D}_{\mathcal{E}} \neq \emptyset$, rather than $\mathfrak{M}_{\mathcal{E}} \neq \emptyset$, that is needed in order to solve the utility maximization problem (4.1).*

The existence of an equivalent supermartingale deflator has some consequences for the class of admissible wealth processes.

Proposition 4.11 *If $\mathfrak{D}_{\mathcal{E}} \neq \emptyset$ then for every $\pi \in \Pi_{\mathcal{E}}$ the wealth process W^π is a semimartingale up to time T (for this concept consult Remark 10.3 in Appendix 2). In particular, $\lim_{t \rightarrow \infty} W_t^\pi$ exists on $\{T = \infty\}$.*

Proof Pick $D \in \mathfrak{D}_{\mathcal{E}}$ and $\pi \in \Pi_{\mathcal{E}}$. Since DW^π is a positive supermartingale, Lemma 10.2 gives that DW^π is a semimartingale up to T . Since D is also a positive supermartingale with $D_T > 0$, $1/D$ is a semimartingale up to T , again by Lemma 10.2. It follows that $W^\pi = (1/D)DW^\pi$ is a semimartingale up to T . \square

In order to complete the discussion, we mention that if a tradeable supermartingale deflator D^* exists, Jensen’s inequality and the supermartingale property of $DW^\rho \equiv D/D^*$ for all $D \in \mathfrak{D}_{\mathcal{E}}$ imply $\mathbb{E}[-\log D_T^*] = \inf_{D \in \mathfrak{D}_{\mathcal{E}}} \mathbb{E}[-\log D_T]$. This can be viewed as an optimality property of the tradeable supermartingale deflator, dual to log-optimality of the numéraire portfolio, as discussed in Sect. 3.7. We can also consider it as a *minimal reverse relative entropy* property of D^* in the class $\mathfrak{D}_{\mathcal{E}}$. Let us explain: for every element $D \in \mathfrak{D}_{\mathcal{E}}$ that is actually a uniformly integrable martingale, consider the probability measure \mathbb{Q} defined by $\mathbb{Q}(A) = \mathbb{E}[D_T \mathbb{1}_A]$; then, the quantity $H(\mathbb{P} | \mathbb{Q}) := \mathbb{E}^{\mathbb{Q}}[D_T^{-1} \log(D_T^{-1})] = \mathbb{E}[-\log D_T]$ is the *relative entropy* of \mathbb{P} with respect to \mathbb{Q} . In general, even when D is not a martingale, we could regard $\mathbb{E}[-\log D_T]$ as the relative entropy of \mathbb{P} with respect to D . The qualifier “reverse” comes from the fact that one usually considers minimizing the entropy of *another* equivalent probability measure \mathbb{Q} with respect to the *original* \mathbb{P} (so-called *minimal entropy measure*). For further details and history we refer the reader to Example 7.1 of Karatzas and Kou [24], and [37] where the minimal reverse relative entropy property of the “minimal martingale measure” for continuous asset-price processes is discussed, as well as

to [19] where a general discussion of minimal distance martingale measures is made (of which the minimal reverse entropy martingale measure is a special case).

4.5 The second main result

Here is our second main result, which places the numéraire portfolio in the context of arbitrage.

Theorem 4.12 *For a financial model described by the stock-price process S and the predictable closed convex constraints \mathcal{C} , the following are equivalent:*

- (1) *The numéraire portfolio exists and $W_T^\rho < \infty$.*
- (2) *The set $\mathcal{D}_{\mathcal{C}}$ of equivalent supermartingale deflators is non-empty.*
- (3) *The NUPBR condition holds.*

The implication (1) \Rightarrow (2) is trivial: $(W^\rho)^{-1}$ is an element of $\mathcal{D}_{\mathcal{C}}$ (observe that we need $W_T^\rho < \infty$ to get $(W_T^\rho)^{-1} > 0$, as required in the definition of $\mathcal{D}_{\mathcal{C}}$).

For the implication (2) \Rightarrow (3), start by assuming that $\mathcal{D}_{\mathcal{C}} \neq \emptyset$ and pick $D \in \mathcal{D}_{\mathcal{C}}$. We wish to show that the collection $(W_T^\pi)_{\pi \in \Pi_{\mathcal{C}}}$, the terminal values of positive wealth processes with $W_0^\pi = 1$ is bounded in probability. Since $D_T > 0$, this is equivalent to showing that the collection $\{D_T W_T^\pi \mid \pi \in \Pi_{\mathcal{C}}\}$ is bounded in probability. But since every process DW^π for $\pi \in \Pi_{\mathcal{C}}$ is a positive supermartingale, we have $\mathbb{P}[D_T W_T^\pi > a] \leq a^{-1} \mathbb{E}[D_T W_T^\pi] \leq a^{-1} \mathbb{E}[D_0 W_0^\pi] = a^{-1}$, for all $a > 0$; this last estimate does not depend on $\pi \in \Pi_{\mathcal{C}}$, and we are done.

Implication (3) \Rightarrow (1) is much harder to prove. One has to analyze what happens when the numéraire portfolio fails to exist; we do this in the next subsection.

Theorem 4.12 provides the equivalent of the FTAP when we only have convex, but not necessarily conic, constraints. Since the existence of a numéraire portfolio ρ with $W_T^\rho < \infty$ is equivalent to $\Psi_T(B, C, \eta) < \infty$, according to Remark 3.16, we obtain also a partial answer to our second question, regarding the characterization of free lunches in terms of predictable characteristics from Sect. 4.3.2; the full answer will be given in the next Sect. 4.6. Finally, the question on utility maximization posed at Sect. 4.3.3 will be tackled in Sect. 4.7.

Remark 4.13 Conditions (2) and (3) of Theorem 4.12 remain invariant by an equivalent change of probability measure. Thus, existence of the numéraire portfolio remains unaffected also, although the numéraire portfolio itself will change. Though a pretty reasonable conjecture to be made at the outset, this does not seem to follow directly from the definition of the numéraire portfolio.

The above fail if we only consider *absolutely continuous* changes of measure (unless S is continuous). One would guess that NUPBR should hold, but non-equivalent changes of probability measure might enlarge the class of admissible wealth processes, since now the positivity condition for wealth processes is more easily satisfied—in effect, the natural constraint set \mathcal{C}_0 can be larger. Consider, for example, a finite time-horizon case where, under \mathbb{P} , X is a driftless compound Poisson process and $\{-1/2, 1/2\}$ is exactly the support of ν . Here, $\mathcal{C}_0 = [-2, 2]$, and X

itself is a martingale. Now consider the simple absolutely continuous change of measure that transforms the jump measure to $\nu_1(dx) := \mathbb{I}_{\{x>0\}}\nu(dx)$; then, $\mathcal{C}_0 = [-2, \infty)$ and, of course, NUIP fails.

Remark 4.14 Theorem 4.12, together with Proposition 4.11, implies that, under NUPBR $_{\mathcal{C}}$, all wealth processes W^π for $\pi \in \Pi_{\mathcal{C}}$ are semimartingales up to infinity. Thus, under NUPBR $_{\mathcal{C}}$, the assumption about the existence of $\lim_{t \rightarrow \infty} W_t^\pi$ on $\{T = \infty\}$, needed for the NA, and the NFLVR conditions in Definition 4.1 are superfluous.

4.6 Consequences of non-existence of the numéraire portfolio

In order to finish the proof of Theorem 4.12, we need to describe *what goes wrong when the numéraire portfolio fails to exist*. This can happen in two ways. Firstly, the set $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ may not have zero $\mathbb{P} \otimes G$ -measure; in this case, Proposition 3.10 shows that one can construct an unbounded increasing profit, the most egregious form of arbitrage. Secondly, when $(\mathbb{P} \otimes G)(\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}) = 0$, the constructed predictable process ρ can fail to be X -integrable (up to time T). The next definition prepares the ground for Proposition 4.16, which describes what happens in this latter case.

Definition 4.15 Consider a sequence $(f_n)_{n \in \mathbb{N}}$ of random variables. Its *limit superior in the probability sense*, $\mathbb{P}\text{-}\limsup_{n \rightarrow \infty} f_n$, is defined as the essential infimum of the collection $\{g \in \mathcal{F} \mid \lim_{n \rightarrow \infty} \mathbb{P}[f_n \leq g] = 1\}$.

It is obvious that the sequence $(f_n)_{n \in \mathbb{N}}$ of random variables is unbounded in probability if and only if $\mathbb{P}\text{-}\limsup_{n \rightarrow \infty} |f_n| = +\infty$ with positive probability.

Proposition 4.16 Assume that the predictable set $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$ -measure and let ρ be the predictable process constructed in Theorem 3.15. Pick any sequence $(\theta_n)_{n \in \mathbb{N}}$ of $[0, 1]$ -valued predictable processes with $\lim_{n \rightarrow \infty} \theta_n = \mathbb{I}$ holding $\mathbb{P} \otimes G$ -almost everywhere and such that $\rho_n := \theta_n \rho$ has bounded support and is X -integrable for all $n \in \mathbb{N}$. Then $\overline{W}_T^\rho := \mathbb{P}\text{-}\limsup_{n \rightarrow \infty} W_T^{\rho_n}$ is a $(0, +\infty]$ -valued random variable and does not depend on the choice of the sequence $(\theta_n)_{n \in \mathbb{N}}$. On $\{(\psi^\rho \cdot G)_T < +\infty\}$, the random variable \overline{W}_T^ρ is an actual limit in probability and

$$\{\overline{W}_T^\rho = +\infty\} = \{(\psi^\rho \cdot G)_T = +\infty\};$$

in particular, $\mathbb{P}[\overline{W}_T^\rho = +\infty] > 0$ if and only if ρ fails to be X -integrable up to T .

The above result says, in effect, that *closely following a numéraire portfolio which is not X -integrable up to time T , one can make arbitrarily large gains with fixed, positive probability*. There are many ways to choose the sequence $(\theta_n)_{n \in \mathbb{N}}$; a particular example is $\theta_n := \mathbb{I}_{\Sigma_n}$ with $\Sigma_n := \{(\omega, t) \in \llbracket 0, T \wedge n \rrbracket \mid |\rho(\omega, t)| \leq n\}$.

Proposition 4.16 is proved in Sect. 8; it answers in a definitive way the question regarding the description of free lunches in terms of predictable characteristics, raised

in Sect. 4.3.2: When $\text{NUPBR}_{\mathcal{C}}$ fails (equivalently, when the numéraire portfolio fails to exist, or exists but $\mathbb{P}[W_T^\rho = \infty] > 0$), there is a way to *construct* the unbounded profit with bounded risk (UPBR) using knowledge of the triplet of predictable characteristics.

Proof of Theorem 4.12 Assuming Proposition 4.16, we are now in a position to show the implication (3) \Rightarrow (1) of Theorem 4.12 and complete its proof. Suppose that the numéraire portfolio fails to exist. Then we either we have opportunities for unbounded increasing profit, in which case NUPBR certainly fails; or the predictable process ρ of Theorem 3.15 exists but is not X -integrable up to time T , in which case Proposition 4.16 provides an UPBR. \square

Remark 4.17 In the context of Proposition 4.16, suppose that $\{\mathcal{I} \cap \check{\mathcal{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$ -measure. The failure of ρ to be X -integrable up to time T can happen in two ways. Start by defining $\tau := \inf\{t \in [0, T] \mid (\psi^\rho \cdot G)_t = +\infty\}$ and $\tau_n := \inf\{t \in [0, T] \mid (\psi^\rho \cdot G)_t \geq n\}$, $n \in \mathbb{N}$. We consider two cases.

First, suppose $\tau > 0$ and $(\psi^\rho \cdot G)_\tau = +\infty$; then $\tau_n < \tau$ for all $n \in \mathbb{N}$ and $\tau_n \uparrow \tau$. By using the sequence $\rho_n := \rho \mathbb{1}_{[0, \tau_n]}$ it is easy to see that $\lim_{n \rightarrow \infty} W_\tau^{\rho_n} = +\infty$ almost surely—this is because $\{(W_t^\rho)^{-1}, 0 \leq t < \tau\}$ is a supermartingale. An example where this happens in finite time is when the returns process X satisfies $dX_t = (1 - t)^{-1/2} dt + d\beta_t$, where β is a standard one-dimensional Brownian motion. Then $\rho_t = (1 - t)^{-1/2}$ and thus $(\psi^\rho \cdot G)_t = \int_0^t (1 - u)^{-1} du$, which gives $\tau \equiv 1$.

With the notation set-up above, let us now give an example with $(\psi^\rho \cdot G)_\tau < +\infty$. Actually, we shall only time-reverse the example we gave before and show that in this case $\tau \equiv 0$. To wit, take the stock-returns process now to be $dX_t = t^{-1/2} dt + d\beta_t$; then $\rho_t = t^{-1/2}$ and $(\psi^\rho \cdot G)_t = \int_0^t u^{-1} du = +\infty$ for all $t > 0$, so that $\tau = 0$. In this case we cannot invest in ρ as before in a “forward” manner, because it has a “singularity” at $t = 0$, and we cannot take full advantage of it. This is basically what makes the proof of Proposition 4.16 non-trivial.

In the case of a continuous-path semimartingale X without portfolio constraints (as the one described in this example), Delbaen and Schachermayer [11] and Levental and Skorohod [32] show that one can actually create “instant arbitrage”, i.e., a non-constant wealth process that never falls below its initial capital (almost the definition of an increasing unbounded profit, but weaker, since the wealth process is not assumed to be increasing). In the presence of jumps, it is an open question whether one can still construct this instant arbitrage—we could not.

4.7 Application to utility optimization

Here we tackle the question that we raised in Sect. 4.3.3. We show that NUPBR is the minimal condition that allows one to solve the utility maximization problem (4.1).

Remark 4.18 The optimization problem (4.1) makes sense only if $u(w) < \infty$. Since U is concave, if $u(w) < +\infty$ for some $w > 0$ then $u(w) < +\infty$ for all $w > 0$ and u is continuous, concave, and increasing. When $u(w) = \infty$ holds for some (equivalently, all) $w > 0$, there are two cases. Either the supremum in (4.1) is not attained, so there is

no solution; or, in case there exists a portfolio with infinite expected utility, concavity of U implies that there will be infinitely many of them.

We begin with the *negative* result: when $\text{NUPBR}_{\mathcal{C}}$ fails, (4.1) cannot be solved.

Proposition 4.19 *Assume that $\text{NUPBR}_{\mathcal{C}}$ fails. Then, for any utility function U , the corresponding utility maximization problem either does not have a solution, or has infinitely many.*

More precisely: If $U(\infty) = +\infty$ then $u(w) = +\infty$ for all $w > 0$, so we either have no solution (when the supremum is not attained) or infinitely many of them (when the supremum is attained); whereas if $U(\infty) < +\infty$, there is no solution.

Proof Since $\text{NUPBR}_{\mathcal{C}}$ fails, pick an $\epsilon > 0$ and a sequence $(\pi_n)_{n \in \mathbb{N}}$ of elements of $\Pi_{\mathcal{C}}$ such that, with $A_n := \{W_T^{\pi_n} \geq n\}$, we have $\mathbb{P}[A_n] \geq \epsilon$ for all $n \in \mathbb{N}$.

If $U(\infty) = +\infty$ then it is obvious that, for all $w > 0$ and $n \in \mathbb{N}$, we have $u(w) \geq \mathbb{E}[U(wW_T^{\pi_n})] \geq \epsilon U(wn)$; thus, $u(w) = +\infty$ and we obtain the result stated in the proposition in view of Remark 4.18.

Now suppose $U(\infty) < \infty$; then $U(w) \leq u(w) \leq U(\infty) < \infty$ for all $w > 0$. Furthermore, u is also concave, thus continuous. Pick any $w > 0$, suppose that $\pi \in \Pi_{\mathcal{C}}$ is optimal for U with initial capital w , and observe that $u(w + n^{-1}) \geq \mathbb{E}[U(wW_T^{\pi} + n^{-1}W_T^{\pi_n})] \geq \mathbb{E}[U(wW_T^{\pi} + \mathbb{I}_{A_n})]$, as well as

$$U(wW_T^{\pi} + \mathbb{I}_{A_n}) = U(wW_T^{\pi}) \mathbb{I}_{\Omega \setminus A_n} + U(wW_T^{\pi} + 1) \mathbb{I}_{A_n}.$$

Pick $M > 0$ large enough so that $\mathbb{P}[wW_T^{\pi} \leq M] \geq 1 - \epsilon/2$; then, for $0 < y \leq M$ the concavity of U gives $U(y + 1) - U(y) \geq U(M + 1) - U(M) =: b$. Therefore,

$$U(wW_T^{\pi} + 1) \geq (U(wW_T^{\pi}) + b) \mathbb{I}_{\{wW_T^{\pi} \leq M\}} + U(wW_T^{\pi}) \mathbb{I}_{\{wW_T^{\pi} > M\}}.$$

Combining the two previous estimates, we get

$$U(wW_T^{\pi} + \mathbb{I}_{A_n}) \geq U(wW_T^{\pi}) + b \mathbb{I}_{A_n \cap \{wW_T^{\pi} \leq M\}}.$$

Since $\mathbb{P}[A_n] \geq \epsilon$, we get $\mathbb{P}[A_n \cap \{wW_T^{\pi} \leq M\}] \geq \epsilon/2$, and setting $a := b\epsilon/2$ we obtain $u(w + n^{-1}) \geq \mathbb{E}[U(wW_T^{\pi} + \mathbb{I}_{A_n})] \geq \mathbb{E}[U(wW_T^{\pi})] + a = u(w) + a$ for all $n \in \mathbb{N}$, which contradicts the continuity of $u(\cdot)$. □

Having discussed what happens when $\text{NUPBR}_{\mathcal{C}}$ fails, let us now assume that it holds. We shall assume a little more structure on the utility function under consideration, namely, that it is continuously differentiable and satisfies the *Inada conditions* $U'(0) = +\infty$ and $U'(+\infty) = 0$.

The $\text{NUPBR}_{\mathcal{C}}$ condition is equivalent to the existence of a numéraire portfolio ρ . Since all wealth processes become supermartingales when divided by W^{ρ} , we conclude that the change of numéraire that utilizes W^{ρ} as a benchmark produces a market for which the *original* \mathbb{P} is a supermartingale measure (see [12] for this “change of

numéraire” technique). In particular, NFLVR $_{\mathcal{C}}$ holds and the “optional decomposition under convex constraints” results of [17] allow us to write down the *superhedging duality*

$$\inf\{w > 0 \mid \exists \pi \in \Pi_{\mathcal{C}} \text{ with } wW_T^\pi \geq H\} = \sup_{D \in \mathcal{D}_{\mathcal{C}}} \mathbb{E}[D_T H],$$

valid for any positive, \mathcal{F}_T -measurable random variable H . This “bipolar” relationship then implies that the utility optimization problem admits a solution (when its value is finite). We send the reader to the papers [30, 31] for more information.

4.8 A word on the additive model

All the results stated up to now hold also when the stock-price processes S^i are *not necessarily positive* semimartingales. Indeed, suppose that we start with initial prices S_0 , introduce $Y := S - S_0$, and define the admissible (discounted) wealth processes class to be generated by strategies $\theta \in \mathcal{P}(\mathbb{R}^d)$ via $W = 1 + \theta \cdot S = 1 + \theta \cdot Y$, where we force $W > 0, W_- > 0$. Here θ is the *number* of shares of stocks in our portfolio. Then, with $\pi := (1/W_-)\theta$, it follows that we can write $W = \mathcal{E}(\pi \cdot Y)$. We do not necessarily have $\Delta Y > -1$ anymore, but this was never used anywhere; the important thing is that admissibility implies $\pi^\top \Delta Y > -1$. Observe that now π does not have a nice interpretation as it had in the case of the multiplicative model.

A final note on constraints. One choice is to require $\theta \in W_- \mathcal{C}$, which is completely equivalent to $\pi \in \mathcal{C}$. A more natural choice would be to enforce them on investment proportions, i.e., to require $(\theta^i S_-^i / W_-)_{1 \leq i \leq d} \in \mathcal{C}$, in which case we get $\pi \in \widehat{\mathcal{C}}$, where $\widehat{\mathcal{C}} := \{x \in \mathbb{R}^d \mid (x^i S_-^i)_{1 \leq i \leq d} \in \mathcal{C}\}$ is predictable.

5 Proof of Proposition 3.10 on the NUIP condition

5.1 If $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ is $\mathbb{P} \otimes G$ -null then NUIP holds

Let us suppose that π is a portfolio with unbounded increasing profit; we shall show that $\{\mathcal{J} \cap \mathcal{C} \neq \emptyset\}$ is not $\mathbb{P} \otimes G$ -null. By definition, then $\{\pi \in \mathcal{C}\}$ has full $\mathbb{P} \otimes G$ -measure, so we wish to prove that $\{\pi \in \mathcal{J}\}$ has strictly positive $\mathbb{P} \otimes G$ -measure.

Now W^π has to be a non-decreasing process, which means that the same holds for $\pi \cdot X$. We also have $\pi \cdot X \neq 0$ with positive probability. This means that the predictable set $\{\pi \notin \mathfrak{N}\}$ has strictly positive $\mathbb{P} \otimes G$ -measure, and it will suffice to show that properties (1–3) of Definition 3.9 hold $\mathbb{P} \otimes G$ -a.e.

Because $\pi \cdot X$ is increasing, we get $\mathbb{I}_{\{\pi^\top x < 0\}} * \mu = 0$, so that $\nu[\pi^\top x < 0] = 0$, $\mathbb{P} \otimes G$ -a.e. In particular, $\pi \cdot X$ is of finite variation, so we must have $\pi \cdot X^c = 0$, and this translates into $\pi^\top c = 0$, $\mathbb{P} \otimes G$ -a.e. For the same reason, one can decompose

$$\pi \cdot X = (\pi \cdot B - [\pi^\top x \mathbb{I}_{\{|x| \leq 1\}}] * \eta) + [\pi^\top x] * \mu. \tag{5.1}$$

The last term $[\pi^\top x] * \mu$ in this decomposition is a pure-jump increasing process, while for the sum of the terms in parentheses we have from (2.4)

$$\Delta(\pi \cdot B - [\pi^\top x \mathbb{I}_{\{|x| \leq 1\}}] * \eta) = \left(\pi^\top b - \int \pi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx) \right) \Delta G = 0.$$

It follows that the term in parentheses on the right-hand side of (5.1) is the continuous part of $\pi \cdot X$ (when seen as a finite variation process) and thus has to be increasing. This translates into the requirement $\pi^\top b - \int \pi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx) \geq 0$, $\mathbb{P} \otimes G$ -a.e., and ends the proof.

5.2 The set-valued process \mathfrak{J} is predictable

In proving the other half of Proposition 3.10, we need to select a predictable process from the set $\{\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset\}$. For this, we shall have to prove that \mathfrak{J} is a predictable set-valued process; however, \mathfrak{J} is not closed, and closedness of sets is crucial when trying to apply measurable selection results. For this reason we have to go through some technicalities first.

Given a triplet (b, c, ν) of predictable characteristics and $a > 0$, define \mathfrak{J}^a to be the set-valued process such that (1–3) of Definition 3.9 hold, as well as

$$\xi^\top b + \int \frac{\xi^\top x}{1 + \xi^\top x} \mathbb{I}_{\{|x| > 1\}} \nu(dx) \geq \frac{1}{a}. \tag{5.2}$$

The following lemma sets forth properties of these sets that we shall find useful.

Lemma 5.1 *With the previous definition we have:*

- (1) \mathfrak{J}^a is increasing in $a > 0$; we have $\mathfrak{J}^a \subseteq \mathfrak{J}$ and $\mathfrak{J} = \bigcup_{a > 0} \mathfrak{J}^a$. In particular, $\mathfrak{J} \cap \check{\mathfrak{C}} \neq \emptyset$ if and only if $\mathfrak{J}^a \cap \check{\mathfrak{C}} \neq \emptyset$ for all large enough $a > 0$.
- (2) For all $a > 0$, \mathfrak{J}^a takes values in closed and convex subsets of \mathbb{R}^d .

Proof In the course of the proof, we suppress dependence of quantities on (ω, t) .

Because of conditions (1–3) of Definition 3.9, the left-hand side of (5.2) is well-defined (the integrand is positive since $\nu[\xi^\top x < 0] = 0$) and has to be positive. In fact, for $\xi \in \mathfrak{J}$, it has to be *strictly* positive, otherwise $\xi \in \mathfrak{N}$. The fact that \mathfrak{J}^a is increasing for $a > 0$ is trivial, and part (1) of this lemma follows.

For part (2), we show first that \mathfrak{J}^a is closed. Observe that the set $\{\xi \in \mathbb{R}^d \mid \xi^\top c = 0 \text{ and } \nu[\xi^\top x < 0] = 0\}$ is closed in \mathbb{R}^d . For ξ in this last set, $x \mapsto \xi^\top x$ is non-negative for all $x \in \mathbb{R}^d$ in a set of full ν -measure. For a sequence $(\xi_n)_{n \in \mathbb{N}}$ in \mathfrak{J}^a with $\lim_{n \rightarrow \infty} \xi_n = \xi$, Fatou’s lemma gives

$$\int \xi^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx) \leq \liminf_{n \rightarrow \infty} \int \xi_n^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx) \leq \liminf_{n \rightarrow \infty} (\xi_n^\top b) = \xi^\top b,$$

so that ξ satisfies (3) of Definition 3.9 also. The measure $\mathbb{I}_{\{|x| > 1\}} \nu(dx)$ (the “large jumps” part of the Lévy measure ν) is finite, and bounded convergence gives

$$\xi^\top b + \int \frac{\xi^\top x}{1 + \xi^\top x} \mathbb{I}_{\{|x| \geq 1\}} \nu(dx) = \lim_{n \rightarrow \infty} \left\{ \xi_n^\top b + \int \frac{\xi_n^\top x}{1 + \xi_n^\top x} \mathbb{I}_{\{|x| \geq 1\}} \nu(dx) \right\} \geq a^{-1}.$$

This establishes that \mathfrak{J}^a is closed. Convexity follows from the fact that the function $x \mapsto x/(1 + x)$ is concave on $(0, \infty)$. □

In view of $\mathcal{J} = \bigcup_{n \in \mathbb{N}} \mathcal{J}^n$ and Lemma 9.3, in order to prove predictability of \mathcal{J} we only have to prove predictability of \mathcal{J}^a . To this end, we define the following real-valued functions, with arguments in $(\Omega \times \mathbb{R}_+) \times \mathbb{R}^d$ (once again, suppressing their dependence on $(\omega, t) \in \llbracket 0, T \rrbracket$):

$$\begin{aligned} z_1(p) &= p^\top c, & z_2(p) &= \int \frac{((p^\top x)^-)^2}{1 + ((p^\top x)^-)^2} \nu(dx), \\ z_3^n(p) &= p^\top b - \int p^\top x \mathbb{I}_{\{n-1 < |x| \leq 1\}} \nu(dx), & \text{for all } n \in \mathbb{N}, & \text{ and} \\ z_4(p) &= p^\top b + \int \frac{p^\top x}{1 + p^\top x} \mathbb{I}_{\{|x| > 1\}} \nu(dx). \end{aligned}$$

Observe that all these functions are predictably measurable in $(\omega, t) \in \Omega \times \mathbb{R}_+$ and continuous in p (follows from applications of the dominated convergence theorem). In a limiting sense, consider formally $z_3(p) \equiv z_3^\infty(p) = p^\top b - \int p^\top x \mathbb{I}_{\{|x| \leq 1\}} \nu(dx)$; observe though that this function might not be well-defined: both the positive and negative parts of the integrand might have infinite ν -integral. Consider also the sequence $\mathcal{A}_n^a := \{p \in \mathbb{R}^d \mid z_1(p) = 0, z_2(p) = 0, z_3^n(p) \geq 0, z_4(p) \geq a^{-1}\}$ of set-valued processes for $n \in \mathbb{N}$, of which the “infinite” version coincides with \mathcal{J}^a , since $\mathcal{J}^a \equiv \mathcal{A}_\infty^a := \{p \in \mathbb{R}^d \mid z_1(p) = 0, z_2(p) = 0, z_3(p) \geq 0, z_4(p) \geq a^{-1}\}$. Because $z_2(p) = 0$, the function z_3 is well-defined (though not necessarily finite, since it can equal $-\infty$). In any case, for any p with $z_2(p) = 0$ we have $\downarrow -\lim_{n \rightarrow \infty} z_3^n(p) = z_3(p)$; so the sequence $(\mathcal{A}_n^a)_{n \in \mathbb{N}}$ is decreasing, and $\downarrow -\lim_{n \rightarrow \infty} \mathcal{A}_n^a = \mathcal{J}^a$. But each \mathcal{A}_n^a is closed and predictable (refer to Lemmas 9.3 and 9.4), and thus so is \mathcal{J}^a .

Remark 5.2 Since $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\} = \bigcup_{n \in \mathbb{N}} \{\mathcal{J}^n \cap \check{\mathcal{C}} \neq \emptyset\}$ and the random set-valued processes \mathcal{J}^n and $\check{\mathcal{C}}$ are closed and predictable, Appendix 1 shows that the set $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ is predictable.

5.3 NUIP implies that $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ is $\mathbb{P} \otimes G$ -null

We are now ready to finish the proof of Proposition 3.10. Let us suppose that $\{\mathcal{J} \cap \check{\mathcal{C}} \neq \emptyset\}$ is not $\mathbb{P} \otimes G$ -null; we shall construct an unbounded increasing profit.

Since $\mathcal{J} = \bigcup_{n \in \mathbb{N}} (\{p \in \mathbb{R}^d \mid |p| \leq n\} \cap \mathcal{J}^n)$, where \mathcal{J}^n is the set-valued process of Lemma 5.1, there exists $n \in \mathbb{N}$ such that the convex, closed, and predictable set-valued process $\mathcal{B}^n := \{p \in \mathbb{R}^d \mid |p| \leq n\} \cap \mathcal{J}^n \cap \check{\mathcal{C}}$ has $(\mathbb{P} \otimes G)(\{\mathcal{B}^n \neq \emptyset\}) > 0$. From Theorem 9.5, there exists a predictable process π such that $\pi(\omega, t) \in \mathcal{B}^n(\omega, t)$ when $\mathcal{B}^n(\omega, t) \neq \emptyset$, and $\pi(\omega, t) = 0$ if $\mathcal{B}^n(\omega, t) = \emptyset$. This π is bounded, so $\pi \in \Pi_{\mathcal{C}}$. The reasoning of Sect. 5.1, now “in reverse”, gives that $\pi \cdot X$ is non-decreasing; the same is then true of W^π . Thus, we *must* have $\mathbb{P}[W_\infty^\pi > 1] > 0$, otherwise $\pi \cdot X \equiv 0$, which is impossible since $(\mathbb{P} \otimes G)(\{\pi \notin \mathfrak{N}\}) > 0$, by construction.

6 Proof of the main Theorem 3.15

We saw in Lemma 3.5 that if the numéraire portfolio ρ exists it has to satisfy $\text{rel}(\pi \mid \rho) \leq 0$ pointwise, $\mathbb{P} \otimes G$ -a.e. In order to find necessary and sufficient con-

ditions for the existence of a (predictable) process ρ that satisfies this inequality, it makes sense first to consider the corresponding static, deterministic problem.

6.1 The exponential Lévy market case

Lévy processes correspond to constant, deterministic triplets of characteristics with respect to the natural time flow $G(t) = t$, so we shall take in this subsection X to be a Lévy process with deterministic Lévy triplet (b, c, ν) ; this means $B_t = bt$, $C_t = ct$ and $\eta(dt, dx) = \nu(dx) dt$ in the notation of Sect. 2.1. We also take \mathcal{C} to be a closed convex subset of \mathbb{R}^d ; recall that $\mathcal{C} \subseteq \mathcal{C}_0$, where $\mathcal{C}_0 := \{\pi \in \mathbb{R}^d \mid \nu[\pi^\top x < -1] = 0\}$. Note that \mathfrak{J} and \mathcal{C} are here both independent of ω and t .

The following result is the deterministic analogue of Theorem 3.15.

Lemma 6.1 *Let (b, c, ν) be a Lévy triplet and \mathcal{C} a closed convex subset of \mathbb{R}^d . Then the following are equivalent:*

- (1) $\mathfrak{J} \cap \check{\mathcal{C}} = \emptyset$.
- (2) *There exists a unique vector $\rho \in \mathcal{C} \cap \mathfrak{N}^\perp$ with $\nu[\rho^\top x \leq -1] = 0$ such that $\text{rel}(\pi \mid \rho) \leq 0$ holds for all $\pi \in \mathcal{C}$.*

If the Lévy measure ν integrates the log, the vector ρ is given as $\rho = \arg \max_{\pi \in \mathcal{C} \cap \mathfrak{N}^\perp} g(\pi)$. In general, ρ is the limit of the optimizers of a sequence of problems in which ν is replaced by a sequence of approximating measures.

We have already shown that if (1) fails then (2) fails as well (actually, we have argued it for the general semimartingale case; see Remark 3.11). The proof of the implication (1) \Rightarrow (2) is quite long—it can be found in [29], Sect. 4, where free lunches for exponential Lévy models are studied in detail.

6.2 Integrability of the numéraire portfolio

We are close to the proof of our main result. We start with a characterization of X -integrability that the predictable process ρ , our candidate for the numéraire portfolio, must satisfy. The following general result is proved in [7].

Theorem 6.2 *Let X be a d -dimensional semimartingale whose triplet of predictable characteristics is (b, c, ν) , relative to the canonical truncation function and some operational clock G . A process $\rho \in \mathcal{P}(\mathbb{R}^d)$ is X -integrable if and only if $(|\widehat{\psi}_i^\rho| \cdot G)_t < \infty$, $i = 1, 2, 3$, for all $t \in \llbracket 0, T \rrbracket$ holds for the predictable processes*

$$\begin{aligned} \widehat{\psi}_1^\rho &:= \rho^\top c \rho, \quad \widehat{\psi}_2^\rho := \int (1 \wedge |\rho^\top x|^2) \nu(dx), \quad \text{and} \\ \widehat{\psi}_3^\rho &:= \rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}}) \nu(dx). \end{aligned}$$

The process $\widehat{\psi}_1^\rho$ controls the quadratic variation of the continuous martingale part of $\rho \cdot X$; the process $\widehat{\psi}_2^\rho$ controls the quadratic variation of the “small-jump” purely

discontinuous martingale part of $\rho \cdot X$ and the intensity of the “large jumps”; whereas $\widehat{\psi}_3^\rho$ controls the drift term of $\rho \cdot X$ when the large jumps are subtracted (it is actually the drift rate of the bounded-jump part). We use Theorem 6.2 to prove Lemma 6.3 below, which provides a necessary and sufficient condition for X -integrability of the candidate for numéraire portfolio.

Lemma 6.3 *Suppose that ρ is a predictable process with $v[\rho^\top x \leq -1] = 0$ and $\text{rel}(0 \mid \rho) \leq 0$. Then ρ is X -integrable if and only if the condition $(\psi^\rho \cdot G)_t(\omega) < \infty$, for all $(\omega, t) \in \llbracket 0, T \rrbracket$, holds for the increasing, predictable process*

$$\psi^\rho := v[\rho^\top x > 1] + \left| \rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}}) \nu(dx) \right|.$$

Proof We have to show that G -integrability of the positive processes ψ_1^ρ and $|\psi_2^\rho|$ (that add up to ψ^ρ) of (3.5) is necessary and sufficient for G -integrability of the three processes $\widehat{\psi}_i^\rho$, $i = 1, 2, 3$, of Theorem 6.2. According to this last theorem, only the sufficiency has to be proved, since the necessity holds trivially (recall $v[\rho^\top x \leq -1] = 0$). Furthermore, from the same theorem, the sufficiency will be established if we can prove that the predictable processes $\widehat{\psi}_1^\rho$ and $\widehat{\psi}_2^\rho$ are G -integrable (note that $\widehat{\psi}_3^\rho$ is already covered by $\widehat{\psi}_2^\rho$).

Dropping the “ ρ ” superscripts, we embark on proving the G -integrability of $\widehat{\psi}_1$ and $\widehat{\psi}_2$, assuming the G -integrability of ψ_1 and ψ_2 in (3.5). The process $\widehat{\psi}_2$ will certainly be G -integrable, if one can show that the positive process

$$\widetilde{\psi}_2 := \int \frac{(\rho^\top x)^2}{1 + \rho^\top x} \mathbb{I}_{\{|\rho^\top x| \leq 1\}} \nu(dx) + \int \frac{\rho^\top x}{1 + \rho^\top x} \mathbb{I}_{\{\rho^\top x > 1\}} \nu(dx)$$

is G -integrable. Since both $-\text{rel}(0 \mid \rho)$ and $\widehat{\psi}_1$ are positive processes, we get that $\widehat{\psi}_1$ and $\widetilde{\psi}_2$ will certainly be G -integrable, if we can show that $\widehat{\psi}_1 + \widetilde{\psi}_2 - \text{rel}(0 \mid \rho)$ is G -integrable. But this last sum is equal to

$$\rho^\top b + \int \rho^\top x (\mathbb{I}_{\{|x|>1\}} - \mathbb{I}_{\{|\rho^\top x|>1\}}) \nu(dx) + 2 \int \frac{\rho^\top x}{1 + \rho^\top x} \mathbb{I}_{\{\rho^\top x > 1\}} \nu(dx);$$

the sum of the first two terms equals ψ_2 , which is G -integrable, and the last (third) term is G -integrable because $\psi_1 = v[\rho^\top x > 1]$ is. □

In the context of Lemma 6.3, if we wish ρ to be X -integrable up to T and not simply X -integrable, we have to impose $\psi_T^\rho < \infty$. This follows from the equivalent characterization of X -integrability up to T in Theorem 6.2, proved in [7].

Theorem 6.2 should be contrasted with Lemma 6.3, where one does not have to worry about the large negative jumps of $\rho \cdot X$, about the quadratic variation of its continuous martingale part, or about the quadratic variation of its small-jump purely discontinuous parts. This follows exactly because in Lemma 6.3 we assume $v[\rho^\top x \leq -1] = 0$ and $\text{rel}(0 \mid \rho) \leq 0$: there are not many negative jumps (none above unit magnitude), and the drift dominates the quadratic variation.

6.3 Proof of Theorem 3.15

The fact that $\{\mathcal{I} \cap \check{\mathcal{C}} \neq \emptyset\}$ is predictable has been shown in Remark 5.2. The claim (2) follows directly from Lemmas 3.5 and 6.3.

For the claims (1.i–1.iii), suppose that $\{\mathcal{I} \cap \check{\mathcal{C}} \neq \emptyset\}$ has zero $\mathbb{P} \otimes G$ -measure. Set $\Lambda := \{\int \log(1 + |x|) \mathbb{I}_{\{|x|>1\}} \nu(dx) < \infty\}$ —on the predictable set Λ , the random measure ν integrates the log. For all $(\omega, t) \in \{\mathcal{I} \cap \check{\mathcal{C}} = \emptyset\} \cap \Lambda$, according to Lemma 6.1, there exists a (uniquely defined) $\rho(\omega, t) \in \mathbb{R}^d$ with $\rho(\omega, t)^\top \Delta X(\omega, t) > -1$ that satisfies $\text{rel}(\pi \mid \rho) \leq 0$, and $\mathfrak{g}(\rho) = \max_{\pi \in \mathcal{C} \cap \mathfrak{N}^\perp} \mathfrak{g}(\pi)$. We also set $\rho = 0$ on the $(\mathbb{P} \otimes G$ -null) set $\{\mathcal{I} \cap \check{\mathcal{C}} = \emptyset\}$.

If $\{\mathcal{I} \cap \check{\mathcal{C}} = \emptyset\} \cap \Lambda$ has full $\mathbb{P} \otimes G$ -measure, we just have to invoke Theorem 9.5 to conclude that ρ is predictable and we are done.

If $\{\mathcal{I} \cap \check{\mathcal{C}} = \emptyset\} \cap \Lambda$ does not have full $\mathbb{P} \otimes G$ -measure, we still have to worry about the predictable set $\{\mathcal{I} \cap \check{\mathcal{C}} = \emptyset\} \cap (\llbracket 0, T \rrbracket \setminus \Lambda)$. On the last set, we consider an approximating sequence $(\nu_n)_{n \in \mathbb{N}}$, keeping every ν_n predictable (this is easy to do, since we can choose all densities f_n to be deterministic—remember our concrete example $f_n(x) = \mathbb{I}_{\{|x| \leq 1\}} + |x|^{-1/n} \mathbb{I}_{\{|x| > 1\}}$); we get a sequence of processes $(\rho_n)_{n \in \mathbb{N}}$ defined on the whole $\llbracket 0, T \rrbracket$ that take values in $\mathcal{C} \cap \mathfrak{N}^\perp$ and solve the corresponding approximating problems on $\{\mathcal{I} \cap \check{\mathcal{C}} = \emptyset\} \cap (\llbracket 0, T \rrbracket \setminus \Lambda)$. According to Lemma 6.1, $(\rho_n)_{n \in \mathbb{N}}$ will converge pointwise to a process ρ ; this will be predictable (as a pointwise limit of predictable processes) and satisfy $\text{rel}(\pi \mid \rho) \leq 0, \forall \pi \in \mathcal{P}_{\mathcal{C}}$.

Now that we have our candidate ρ for numéraire portfolio, we only need to check its X -integrability; according to Lemma 6.3 this is covered by the criterion $(\phi^\rho \cdot G)_t < +\infty$ for all $t \in \llbracket 0, T \rrbracket$. In light of Lemma 3.5, we are done. \square

7 On rates of convergence to zero for positive supermartingales

Every positive supermartingale converges, as time tends to infinity. The following decides whether this limit is zero or not in terms of predictable characteristics and estimates the rate of convergence to zero when this is the case.

Proposition 7.1 *Let Z be a local supermartingale with $\Delta Z > -1$ and Doob–Meyer decomposition $Z = M - A$, where A is an increasing, predictable process. With $\widehat{C} := [Z^c, Z^c]$ the quadratic covariation of the continuous local martingale part of Z and $\hat{\eta}$ the predictable compensator of the jump measure $\hat{\mu}$, define the increasing predictable process $H := A + \widehat{C}/2 + q(1 + x) * \hat{\eta}$, where $q : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is the convex function $q(y) := [-\log a + (1 - a^{-1})y] \mathbb{I}_{(0,a)}(y) + [y - 1 - \log y] \mathbb{I}_{[a,+\infty)}(y)$ for some $a \in (0, 1)$.*

Consider also the positive supermartingale $Y = \mathcal{E}(Z)$. Then on the event $\{H_\infty < +\infty\}$ we have $\lim_{t \rightarrow \infty} Y_t \in (0, +\infty)$, while on $\{H_\infty = +\infty\}$ we have $\limsup_{t \rightarrow \infty} (H_t^{-1} \log Y_t) \leq -1$.

Proposition 7.1 is an abstract version of Proposition 3.21; to obtain that latter proposition from the former, notice that W^π / W^ρ is a positive supermartingale, and

identify the elements A, \widehat{C} and $q(1+x) * \hat{\eta}$ of Proposition 7.1 with $\text{rel}(\pi \mid \rho) \cdot G, (\pi - \rho)^\top c(\pi - \rho) \cdot G$, and $(\int q_a(\frac{1+\pi^\top x}{1+\rho^\top x})v(dx)) \cdot G$.

If we further assume $\Delta Z \geq -1 + \delta$ for some $\delta > 0$ then, by considering $q(x) = x - \log(1+x)$ in the definition of H , we obtain $\lim_{t \rightarrow \infty} (H_t^{-1} \log Y_t) = -1$ on the set $\{H_\infty = +\infty\}$, i.e., we get the exact rate of decay of $\log Y$ to $-\infty$.

Remark 7.2 In the course of the proof, we shall make heavy use of the following: For a locally square integrable martingale N with angle-bracket (predictable quadratic variation) process $\langle N, N \rangle$, on the event $\{\langle N, N \rangle_\infty < +\infty\}$ the limit N_∞ exists and is finite, whereas on the event $\{\langle N, N \rangle_\infty = +\infty\}$ we have $\lim_{t \rightarrow \infty} N_t / \langle N, N \rangle_t = 0$.

Note also that if $N = v(x) * (\hat{\mu} - \hat{\eta})$ then $\langle N, N \rangle \leq v(x)^2 * \hat{\eta}$ (equality holds if and only if N is quasi-left-continuous). Combining this with the previous remarks we get that on the event $\{(v(x)^2 * \hat{\eta})_\infty < +\infty\}$ the limit N_∞ exists and is finite, whereas on $\{(v(x)^2 * \hat{\eta})_\infty = +\infty\}$ we have $\lim_{t \rightarrow \infty} N_t / (v(x)^2 * \hat{\eta})_t = 0$.

Proof of Proposition 7.1 For the supermartingale $Y = \mathcal{E}(Z)$, the stochastic exponential formula (1.1) gives $\log Y = Z - [Z^c, Z^c]/2 - \sum_{s \leq \cdot} [\Delta Z_s - \log(1 + \Delta Z_s)]$ or, equivalently,

$$\log Y = -A + (M^c - \widehat{C}/2) + (x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu}). \tag{7.1}$$

We start with the continuous local martingale part, and use Remark 7.2 twice: first, on $\{\widehat{C}_\infty < +\infty\}$, M_∞^c exists and is real-valued; second, on $\{\widehat{C}_\infty = +\infty\}$, we get $\lim_{t \rightarrow \infty} (M_t^c - \widehat{C}_t/2) / (\widehat{C}_t/2) = -1$.

To deal with the purely discontinuous local martingale part, we first define the two indicator functions $\ell := \mathbb{I}_{[-1, -1+a)}$ and $r := \mathbb{I}_{[-1+a, +\infty)}$, where ℓ and r stand as mnemonics for *l*eft and *r*ight. Define the two semimartingales

$$\begin{aligned} E &:= [\ell(x) \log(1+x)] * \hat{\mu} - [\ell(x)x] * \hat{\eta}, \\ F &:= [r(x) \log(1+x)] * (\hat{\mu} - \hat{\eta}) + [r(x)q(1+x)] * \hat{\eta}, \end{aligned}$$

and observe that $x * (\hat{\mu} - \hat{\eta}) - [x - \log(1+x)] * \hat{\mu} = E + F$.

We claim that, on $\{(q(1+x) * \hat{\eta})_\infty < +\infty\}$, both E_∞ and F_∞ exist and are real-valued. For E , this happens because $([\ell(x)q(1+x)] * \hat{\eta})_\infty < +\infty$ implies that there will only be a finite number of times when $\Delta Z \in (-1, -1+a]$ so that both terms in the definition of E have a limit at infinity. Turning to F , the second term in its definition is, obviously, finite-valued at infinity, whereas for the local martingale term $[r(x) \log(1+x)] * (\hat{\mu} - \hat{\eta})$ we need only use the set inclusion $\{([r(x)q(1+x)] * \hat{\eta})_\infty < +\infty\} \subseteq \{([r(x) \log^2(1+x)] * \hat{\eta})_\infty < +\infty\}$ to get that it has finite predictable quadratic variation and use Remark 7.2.

Now we turn attention to the event $\{(q(1+x) * \hat{\eta})_\infty = +\infty\}$; there at least one of the quantities $([\ell(x)q(1+x)] * \hat{\eta})_\infty$ and $([r(x)q(1+x)] * \hat{\eta})_\infty$ must be infinite.

On the event $\{([r(x)q(1+x)] * \hat{\eta})_\infty = \infty\}$, use of the definition of F and then Remark 7.2 gives $\lim_{t \rightarrow \infty} F_t / ([r(x)q(1+x)] * \hat{\eta})_t = -1$.

Now let us work on the event $\{([\ell(x)q(1+x)] * \hat{\eta})_\infty = \infty\}$. We know that the inequality $\log y \leq y - 1 - q(y)$ holds for $y > 0$; using this last inequality in the first

term in the definition of E we get $E \leq [\ell(x)(x - q(1 + x))] * \hat{\mu} - [\ell(x)x] * \hat{\eta}$, or, further, that $E \leq [\ell(x)(x - q(1 + x))] * (\hat{\mu} - \hat{\eta}) - [\ell(x)q(1 + x)] * \hat{\eta}$. From this last inequality and Remark 7.2 we get $\limsup_{t \rightarrow \infty} E_t / ([\ell(x)q(1 + x)] * \hat{\eta})_t \leq -1$.

Let us summarize the last paragraphs on the purely discontinuous part. On the event $\{(q(1 + x) * \hat{\eta})_\infty < +\infty\}$, the limit $(x * (\hat{\mu} - \hat{\eta}) - [x - \log(1 + x)] * \hat{\mu})_\infty$ exists and is finite; on the other hand, on the event $\{(q(1 + x) * \hat{\eta})_\infty = +\infty\}$, we have $\limsup_{t \rightarrow \infty} (x * (\hat{\mu} - \hat{\eta}) - [x - \log(1 + x)] * \hat{\mu})_t / (q(1 + x) * \hat{\eta})_t \leq -1$.

From the previous discussion on the continuous and the purely discontinuous local martingale parts of $\log Y$ and the definition of H , the result follows. □

8 Proof of Proposition 4.16

8.1 The proof

Start by defining $\Omega_0 := \{(\psi^\rho \cdot G)_T < \infty\}$ and $\Omega_A := \Omega \setminus \Omega_0$.

First, we show the result for Ω_0 . Assume $\mathbb{P}[\Omega_0] > 0$, and call \mathbb{P}_0 the probability measure one gets by conditioning \mathbb{P} on the set Ω_0 . The process ρ , of course, remains predictable when viewed under the new measure; and, because we are restricting ourselves to Ω_0 , ρ is X -integrable up to T under \mathbb{P}_0 .

By a simple use of the dominated convergence theorems for Lebesgue and for stochastic integrals, all three sequences of processes $\rho_n \cdot X$, $[\rho_n \cdot X^c, \rho_n \cdot X^c]$ and $\sum_{s \leq \cdot} [\rho_n^\top \Delta X_s - \log(1 + \rho_n^\top \Delta X_s)]$ converge uniformly (in $t \in [0, T]$) in \mathbb{P}_0 -measure to three processes, that do not depend on the sequence $(\rho_n)_{n \in \mathbb{N}}$. Then the stochastic exponential formula (1.1) gives that $W_T^{\rho_n}$ converges in \mathbb{P}_0 -measure to a random variable, which does not depend on the sequence $(\rho_n)_{n \in \mathbb{N}}$. Since the limit of the sequence $(\mathbb{I}_{\Omega_0} W_T^{\rho_n})_{n \in \mathbb{N}}$ is the same under both the \mathbb{P} -measure and the \mathbb{P}_0 -measure, we conclude that, on Ω_0 , the sequence $(W_T^{\rho_n})_{n \in \mathbb{N}}$ converges in \mathbb{P} -measure to a real-valued random variable, independently of the choice of the sequence $(\rho_n)_{n \in \mathbb{N}}$.

Now we have to tackle the set Ω_A , which is trickier. We shall use a “helping sequence of portfolios”. Suppose $\mathbb{P}[\Omega_A] > 0$; otherwise, there is nothing to prove. Under this assumption, there exists a sequence of $[0, 1]$ -valued predictable processes $(h_n)_{n \in \mathbb{N}}$ such that each $\pi_n := h_n \rho$ is X -integrable up to T and the sequence of terminal values $((\pi_n \cdot X)_T)_{n \in \mathbb{N}}$ is unbounded in probability (readers unfamiliar with this fact should consult [5], Corollary 3.6.10, p. 128). It is reasonable to believe (but wrong in general, and a little tedious to show in our case) that unboundedness in probability of the terminal values $((\pi_n \cdot X)_T)_{n \in \mathbb{N}}$ implies that the sequence of the terminal values for the stochastic exponentials $(W_T^{\pi_n})_{n \in \mathbb{N}}$ is also unbounded in probability. We shall show this in Lemma 8.1 of the next subsection; for the time being, we accept this as a fact. Then $\mathbb{P}[\limsup_{n \rightarrow \infty} W_T^{\pi_n} = +\infty] > 0$, where the \limsup is taken in probability and not almost surely (recall Definition 4.15).

Let us return to our original sequence of portfolios $(\rho_n)_{n \in \mathbb{N}}$ with $\rho_n = \theta_n \rho$ and show that $\{\limsup_{n \rightarrow \infty} W_T^{\pi_n} = +\infty\} \subseteq \{\limsup_{n \rightarrow \infty} W_T^{\rho_n} = +\infty\}$. Both of these upper limits and, in fact, all the \limsup that will appear until the end of the proof are supposed to be in \mathbb{P} -measure. Since each θ_n is $[0, 1]$ -valued and $\lim_{n \rightarrow \infty} \theta_n = \mathbb{I}$, one can choose an increasing sequence $(k(n))_{n \in \mathbb{N}}$ of natural numbers such that the sequence $(W_T^{\theta_{k(n)} \pi_n})_{n \in \mathbb{N}}$ is unbounded in \mathbb{P} -measure on the set

$\{\limsup_{n \rightarrow \infty} W_T^{\pi_n} = +\infty\}$. Now each process $W^{\theta_{k(n)}\pi_n} / W^{\rho_{k(n)}}$ is a positive supermartingale, since $\text{rel}(\theta_{k(n)}\pi_n \mid \rho_{k(n)}) = \text{rel}(\theta_{k(n)}h_n\rho \mid h_n\rho) \leq 0$, the last inequality due to the fact that $[0, 1] \ni u \mapsto \mathbf{g}(u\rho)$ is increasing, and so the sequence of random variables $(W_T^{\theta_{k(n)}\pi_n} / W_T^{\rho_{k(n)}})_{n \in \mathbb{N}}$ is bounded in probability. From the last two facts follows that the sequence of random variables $(W_T^{\pi_n})_{n \in \mathbb{N}}$ is also unbounded in \mathbb{P} -measure on $\{\limsup_{n \rightarrow \infty} W_T^{\pi_n} = +\infty\}$.

Up to now we have shown $\mathbb{P}[\limsup_{n \rightarrow \infty} W_T^{\rho_n} = +\infty] > 0$ and we also know $\{\limsup_{n \rightarrow \infty} W_T^{\rho_n} = +\infty\} \subseteq \Omega_A$; it remains to show that the last set inclusion is actually an equality (mod \mathbb{P}). Set $\Omega_B := \Omega_A \setminus \{\limsup_{n \rightarrow \infty} W_T^{\rho_n} = +\infty\}$ and assume that $\mathbb{P}[\Omega_B] > 0$. Working under the conditional measure on Ω_B (denoted by \mathbb{P}_B) and following the exact same steps we carried out two paragraphs ago, we find predictable processes $(h_n)_{n \in \mathbb{N}}$ such that each $\pi_n := h_n\rho$ is X -integrable up to T under \mathbb{P}_B and such that the sequence of terminal values $((\pi_n \cdot X)_{\infty})_{n \in \mathbb{N}}$ is unbounded in \mathbb{P}_B -probability; then $\mathbb{P}_B[\limsup_{n \rightarrow \infty} W_T^{\rho_n} = +\infty] > 0$, which contradicts the definition of Ω_B , and we are done. □

8.2 Unboundedness for stochastic exponentials

We still owe one thing in the previous proof: at some point we had a sequence of random variables $((\pi_n \cdot X)_T)_{n \in \mathbb{N}}$ that was unbounded in probability and needed to show that the sequence $(\mathcal{E}(\pi_n \cdot X)_T)_{n \in \mathbb{N}}$ is unbounded in probability as well. One has to be careful with statements like that because, as we shall see in Remark 8.2, the stochastic—unlike the usual—exponential is *not* a monotone operation.

We have to prove the following Lemma 8.1 and finish the proof of Proposition 4.16. To begin, observe that with $R_n := \pi_n \cdot X$ the collection $(R_n)_{n \in \mathbb{N}}$ is such that $\Delta R_n > -1$, and $\mathcal{E}(R_n)^{-1}$ is a positive supermartingale for all $n \in \mathbb{N}$.

A class \mathcal{R} of semimartingales will be called “unbounded in probability” if the collection $\{\sup_{t \in [0, T]} |R_t| \mid R \in \mathcal{R}\}$ is unbounded in probability. Similar definitions apply for (un)boundedness from above and below, taking one-sided suprema.

Lemma 8.1 *Let \mathcal{R} be a collection of semimartingales such that $R_0 = 0$, $\Delta R > -1$ and suppose $\mathcal{E}(R)^{-1}$ is a (positive) supermartingale for all $R \in \mathcal{R}$ (in particular, $\mathcal{E}(R)_T$ exists and takes values in $(0, \infty)$). Then the collection of processes \mathcal{R} is unbounded in probability if and only if the collection of positive random variables $\{\mathcal{E}(R)_T \mid R \in \mathcal{R}\}$ is unbounded in probability.*

Proof We shall only consider boundedness notions “in probability” throughout. Since $R \geq \log \mathcal{E}(R)$ for all $R \in \mathcal{R}$, one side of the equivalence is trivial, and we only have to prove that if \mathcal{R} is unbounded then $\{\mathcal{E}(R)_T \mid R \in \mathcal{R}\}$ is unbounded. We split the proof of this into four steps.

(i) Since $\{\mathcal{E}(R)^{-1} \mid R \in \mathcal{R}\}$ is a collection of positive supermartingales, it is bounded from above, thus $\{\log \mathcal{E}(R) \mid R \in \mathcal{R}\}$ is bounded from below. Since $R \geq \log \mathcal{E}(R)$ for all $R \in \mathcal{R}$ and \mathcal{R} is unbounded, it follows that it *must* be unbounded from above.

(ii) Let us now show that *the collection of random variables $\{\mathcal{E}(R)_T \mid R \in \mathcal{R}\}$ is unbounded if and only if the collection of semimartingales $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ is*

unbounded (from above, of course, since they are positive). One direction is trivial: if the semimartingale class is unbounded, the random variable class is unbounded, too; we only need to argue the reverse implication. Unboundedness of $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ means that we can pick an $\epsilon > 0$ so that, for any $n \in \mathbb{N}$, there exists a semimartingale $R^n \in \mathcal{R}$ such that for the stopping times $\tau_n := \inf\{t \in [0, T] \mid \mathcal{E}(R^n)_t \geq n\}$ (as usual, we set $\tau_n = \infty$ where the last set is empty) we have $\mathbb{P}[\tau_n < \infty] \geq \epsilon$. Each $\mathcal{E}(R^n)^{-1}$ is a supermartingale; therefore,

$$\mathbb{P}[\mathcal{E}(R^n)_T^{-1} \leq n^{-1/2}] \geq \mathbb{P}[\mathcal{E}(R^n)_T^{-1} \leq n^{-1/2} \mid \tau_n < \infty] \mathbb{P}[\tau_n < \infty] \geq \epsilon(1 - n^{-1/2}),$$

so $(\mathcal{E}(R^n)_T)_{n \in \mathbb{N}}$ is unbounded, and the claim of this paragraph is proved.

We want to show now that if \mathcal{R} is unbounded then $\{\mathcal{E}(R) \mid R \in \mathcal{R}\}$ is unbounded, too. Define the class $\mathcal{Z} := \{\mathcal{L}(\mathcal{E}(R)^{-1}) \mid R \in \mathcal{R}\}$; we have $Z_0 = 0$, $\Delta Z > -1$, and that Z is a local supermartingale for all $Z \in \mathcal{Z}$.

(iii) Let us prove that *if the collection \mathcal{Z} is bounded from below then it is also bounded from above*. To this end, pick any $\epsilon > 0$. We can find an $M \in \mathbb{R}_+$ such that the stopping times $\tau_Z := \inf\{t \in [0, T] \mid Z_t \leq -M + 1\}$ (we set $\tau_Z = \infty$ where the last set is empty) satisfy $\mathbb{P}[\tau_Z < \infty] \leq \epsilon/2$ for all $Z \in \mathcal{Z}$. Since $\Delta Z > -1$, we have $Z_{\tau_Z} \geq -M$, and so each stopped process Z^{τ_Z} is a supermartingale (it is a local supermartingale bounded uniformly from below). Then, with $y_\epsilon := 2M/\epsilon$, we have

$$\mathbb{P}\left[\sup_{t \in [0, T]} Z_t > y_\epsilon\right] \leq (\epsilon/2) + \mathbb{P}\left[\sup_{t \in [0, T]} Z_t^{\tau_Z} > y_\epsilon\right] \leq (\epsilon/2) + (1 + y_\epsilon/M)^{-1} \leq \epsilon,$$

and thus \mathcal{Z} is bounded from above, too

(iv) Now we have all the ingredients for the proof. Suppose that \mathcal{R} is unbounded; we have seen that it has to be unbounded from above. Using Lemma 3.4 with $Y \equiv 0$, we get that every $Z \in \mathcal{Z}$ is of the form

$$Z = -R + [R^c, R^c] + \sum_{s \leq \cdot} \frac{|\Delta R_s|^2}{1 + \Delta R_s}. \tag{8.1}$$

When \mathcal{Z} is unbounded from below things are pretty simple, because $\log \mathcal{E}(Z) \leq Z$ for all $Z \in \mathcal{Z}$ so that $\{\log \mathcal{E}(Z) \mid Z \in \mathcal{Z}\}$ is unbounded from below and, thus, $\{\mathcal{E}(R) \mid R \in \mathcal{R}\} = \{\exp(-\log \mathcal{E}(Z)) \mid Z \in \mathcal{Z}\}$ is unbounded from above.

It remains to see what happens if \mathcal{Z} is bounded from below. From step (iii) we know that \mathcal{Z} must be bounded from above as well. Then, because of (3.1) and the unboundedness from above of \mathcal{R} , this would mean that the collection $\{[R^c, R^c] + \sum_{s \leq \cdot} [|\Delta R_s|^2 / (1 + \Delta R_s)] \mid R \in \mathcal{R}\}$ of increasing processes is also unbounded. Now for $Z \in \mathcal{Z}$ we have

$$\log \mathcal{E}(Z) = -\log \mathcal{E}(R) = -R + \frac{1}{2}[R^c, R^c] + \sum_{s \leq \cdot} [\Delta R_s - \log(1 + \Delta R_s)]$$

from (8.1) and the stochastic exponential formula, so that

$$Z - \log \mathcal{E}(Z) = \frac{1}{2}[R^c, R^c] + \sum_{s \leq \cdot} \left[\log(1 + \Delta R_s) - \frac{\Delta R_s}{1 + \Delta R_s} \right].$$

The collection of increasing processes on the right-hand side of this last equation is unbounded, because $\{[R^c, R^c] + \sum_{s \leq \cdot} [(\Delta R_s)^2 / (1 + \Delta R_s)] \mid R \in \mathcal{R}\}$ is unbounded, too; as we observed. But since \mathcal{Z} is bounded, this means that $\{\log \mathcal{E}(\mathcal{Z}) \mid Z \in \mathcal{Z}\}$ is unbounded from below, and we conclude again as before. \square

Remark 8.2 Without the assumption that $\{\mathcal{E}(R)^{-1} \mid R \in \mathcal{R}\}$ consists of supermartingales, this result is no longer true. In fact, take $T \equiv +\infty$ and $\mathcal{R} = \{R\}$ where $R_t = at + \beta_t$, with $a \in (0, 1/2)$ and β is a standard one-dimensional Brownian motion. Then R is bounded from below and unbounded from above; nevertheless, $\log \mathcal{E}(R)_t = (a - 1/2)t + \beta_t$ is bounded from above and unbounded from below.

Acknowledgements Results in this paper are drawn in part from the second author’s doctoral dissertation [28]. Work partially supported by the National Science Foundation, under grant NSF-DMS-06-01774. The authors would like to express their deep gratitude towards the editor and two anonymous referees for their invaluable input and comments, as well as their thoroughness while going through the lengthy manuscript.

Appendix 1 Measurable random subsets

Throughout this section we shall be working on a measurable space $(\tilde{\Omega}, \mathcal{P})$; although the results are general, think of $\tilde{\Omega}$ as $\Omega \times \mathbb{R}_+$ and of \mathcal{P} as the *predictable* σ -algebra. The metric of the Euclidean space \mathbb{R}^d is denoted by “dist” and its generic point by z . Proofs of the results below will not be given, but can be found (in greater generality) in Chap. 17 of [2]; for shorter proofs of the specific results, see [28]. The subject of measurable random subsets and measurable selection is slightly gory in its technicalities, but the statements should be intuitively clear.

A *random subset* of \mathbb{R}^d is just a random variable taking values in $2^{\mathbb{R}^d}$, the powerset (class of all subsets) of \mathbb{R}^d . Thus, a random subset of \mathbb{R}^d is a function $\mathfrak{A} : \tilde{\Omega} \mapsto 2^{\mathbb{R}^d}$. A random subset \mathfrak{A} of \mathbb{R}^d will be called *closed* (resp., *convex*) if the set $\mathfrak{A}(\tilde{\omega})$ is closed (resp., convex) for every $\tilde{\omega} \in \tilde{\Omega}$.

Measurability requirements on random subsets are given by placing some measurable structure on the space $2^{\mathbb{R}^d}$, which we endow with the smallest σ -algebra that makes the mappings $2^{\mathbb{R}^d} \ni A \mapsto \text{dist}(z, A) \in \mathbb{R}_+ \cup \{+\infty\}$ measurable for all $z \in \mathbb{R}^d$ (by definition, $\text{dist}(z, \emptyset) = +\infty$). The following equivalent formulations are sometimes useful.

Proposition 9.1 *The constructed σ -algebra on $2^{\mathbb{R}^d}$ is also the smallest σ -algebra that makes the class $\{2^{\mathbb{R}^d} \ni A \mapsto \mathbb{I}_{\{A \cap K \neq \emptyset\}}\}$, for every compact (resp. closed, resp. open) $K \subseteq \mathbb{R}^d$ of functions measurable.*

From Proposition 9.1, a random subset \mathfrak{A} of \mathbb{R}^d is measurable if for any compact $K \subseteq \mathbb{R}^d$, the set $\{\mathfrak{A} \cap K \neq \emptyset\} := \{\tilde{\omega} \in \tilde{\Omega} \mid \mathfrak{A}(\tilde{\omega}) \cap K \neq \emptyset\}$ is \mathcal{P} -measurable.

Remark 9.2 Suppose that the random subset \mathfrak{A} is a *singleton* $\mathfrak{A}(\tilde{\omega}) = \{a(\tilde{\omega})\}$ for some $a : \tilde{\Omega} \mapsto \mathbb{R}^d$. Then \mathfrak{A} is measurable if and only if $\{a \in K\} \in \mathcal{P}$ for all closed $K \subseteq \mathbb{R}^d$, i.e., if and only if a is \mathcal{P} -measurable.

We now deal with unions and intersections of random subsets of \mathbb{R}^d .

Lemma 9.3 *Suppose that $(\mathcal{A}_n)_{n \in \mathbb{N}}$ is a sequence of measurable random subsets of \mathbb{R}^d . Then the union $\bigcup_{n \in \mathbb{N}} \mathcal{A}_n$ is also measurable. If, furthermore, each random subset \mathcal{A}_n is closed then the intersection $\bigcap_{n \in \mathbb{N}} \mathcal{A}_n$ is measurable.*

The following lemma gives a way to construct measurable, closed random subsets of \mathbb{R}^d . To state it, we shall need (a slight generalization of) the notion of Carathéodory function. For a measurable closed random subset \mathcal{A} of \mathbb{R}^d , a mapping f of $\tilde{\Omega} \times \mathbb{R}^d$ into another topological space will be called *Carathéodory on \mathcal{A}* , if it is measurable (with respect to the product σ -algebra on $\tilde{\Omega} \times \mathbb{R}^d$), and if $z \mapsto f(\tilde{\omega}, z)$ is continuous on $\mathcal{A}(\tilde{\omega})$, for each $\tilde{\omega} \in \tilde{\Omega}$. Of course, if $\mathcal{A} \equiv \mathbb{R}^d$, we recover the usual textbook notion of a Carathéodory function.

Lemma 9.4 *Let E be any topological space, $F \subseteq E$ a closed subset, and \mathcal{A} a closed and convex random subset of \mathbb{R}^d . If $f : \tilde{\Omega} \times \mathbb{R}^d \rightarrow E$ is a Carathéodory function on \mathcal{A} then $\mathcal{C} := \{z \in \mathcal{A} \mid f(\cdot, z) \in F\}$ is closed and measurable.*

The last result focuses on the measurability of the “argument” process in random optimization problems.

Theorem 9.5 *Suppose that \mathcal{A} is a closed and convex, measurable, non-empty random subset of \mathbb{R}^d , and $f : \tilde{\Omega} \times \mathbb{R}^d \mapsto \mathbb{R} \cup \{-\infty\}$ is a Carathéodory function on \mathcal{A} . For the optimization problem $f_*(\tilde{\omega}) = \sup_{z \in \mathcal{A}} f(\tilde{\omega}, z)$, we have:*

- (1) *The value function f_* is \mathcal{P} -measurable.*
- (2) *Suppose that $f_*(\tilde{\omega})$ is finite for all $\tilde{\omega}$, and that there exists a unique $z_*(\tilde{\omega}) \in \mathcal{A}(\tilde{\omega})$ satisfying $f(\tilde{\omega}, z_*(\tilde{\omega})) = f_*(\tilde{\omega})$. Then $\tilde{\omega} \mapsto z_*(\tilde{\omega})$ is \mathcal{P} -measurable.*

In particular, if \mathcal{A} is a closed and convex, measurable, non-empty random subset of \mathbb{R}^d , we can find a \mathcal{P} -measurable $h : \tilde{\Omega} \rightarrow \mathbb{R}^d$ with $h(\tilde{\omega}) \in \mathcal{A}(\tilde{\omega})$ for all $\tilde{\omega} \in \tilde{\Omega}$.

For the “particular” case of the last theorem one can use, for example, the function $f(x) = -|x|$ and the result of the first part of the theorem.

Appendix 2 Semimartingales and stochastic integration up to $+\infty$

We recall here a few important concepts from [7] and prove a few useful results. One can also check [9] for the ideas presented below.

Definition 10.1 Let $X = (X_t)_{t \in \mathbb{R}_+}$ be a semimartingale such that $X_\infty := \lim_{t \rightarrow \infty} X_t$ exists. Then X will be called a *semimartingale up to infinity* if the process \tilde{X} defined on the time interval $[0, 1]$ by $\tilde{X}(t) = X(t/(1-t))$ (of course, $\tilde{X}_1 = X_\infty$) is a semimartingale relative to the filtration $\tilde{\mathbb{F}} = (\tilde{\mathcal{F}}_t)_{t \in [0,1]}$ defined by $\tilde{\mathcal{F}}_t := \mathcal{F}_{t/(1-t)}$ for $0 \leq t < 1$ and $\tilde{\mathcal{F}}_1 := \bigvee_{t \in \mathbb{R}_+} \mathcal{F}_t$.

Similarly, we define local martingales up to infinity, processes of finite variation up to infinity, etc., if the corresponding process \tilde{X} has the property.

Fix a d -dimensional semimartingale X . An X -integrable predictable process π will be called X -integrable up to infinity if $\pi \cdot X$ is a semimartingale up to infinity.

To appreciate the difference between a semimartingale with limit at infinity and a semimartingale up to infinity, consider the simple example where X is the deterministic, continuous process $X_t := t^{-1} \sin t$; then X is a semimartingale with $X_\infty = 0$, but $\text{Var}(X)_\infty = +\infty$, and thus X cannot be a semimartingale up to infinity (a deterministic semimartingale must be of finite variation).

Every semimartingale up to infinity X can be written as a sum $X = A + M$, where A is a process of finite variation up to infinity (which simply means that $\text{Var}(A)_\infty < \infty$) and M is a local martingale up to infinity (which means that there exists an increasing sequence of stopping times $(T_n)_{n \in \mathbb{N}}$ with $\{T_n = +\infty\} \uparrow \Omega$ such that each of the stopped processes M^{T_n} is a uniformly integrable martingale).

Lemma 10.2 *A positive supermartingale Z is a special semimartingale up to infinity. If, furthermore, $Z_\infty > 0$ then $\mathcal{L}(Z)$ is also a special semimartingale up to infinity, and both Z^{-1} and $\mathcal{L}(Z^{-1})$ are semimartingales up to infinity.*

Proof We start with the Doob–Meyer decomposition $Z = M - A$, where M is a local martingale with $M_0 = Z_0$ and A is an increasing, predictable process. The positive local martingale M is a supermartingale, and we can infer that both limits Z_∞ and M_∞ exist and are integrable. This means that A_∞ exists and, actually, $\mathbb{E}[A_\infty] = \mathbb{E}[M_\infty] - \mathbb{E}[Z_\infty] < \infty$, so A is a predictable process of integrable variation up to infinity. It remains to show that M is a local martingale up to infinity. Set $T_n := \inf\{t \geq 0 \mid M_t \geq n\}$; this obviously satisfies $\{T_n = +\infty\} \uparrow \Omega$ (the supremum of a positive supermartingale is finite). Since $\sup_{0 \leq t \leq T_n} M_t \leq n + M_{T_n} \mathbb{1}_{\{T_n < \infty\}}$ and, by the optional sampling theorem $\mathbb{E}[M_{T_n} \mathbb{1}_{\{T_n < \infty\}}] \leq \mathbb{E}[M_0] < \infty$, we get $\mathbb{E}[\sup_{0 \leq t \leq T_n} M_t] < \infty$. Thus, the local martingale M^{T_n} is actually a uniformly integrable martingale, and thus Z is a special semimartingale up to infinity.

Now assume that $Z_\infty > 0$. Since Z is a supermartingale, this will mean that both \tilde{Z} and \tilde{Z}_- are bounded away from zero. (A “tilde” over a process means that we are considering the process of Definition 10.1 under the new filtration $\tilde{\mathbf{F}}$.) Since \tilde{Z}_-^{-1} is locally bounded and \tilde{Z} is a special semimartingale, $\mathcal{L}(\tilde{Z}) = \tilde{Z}_-^{-1} \cdot \tilde{Z}$ will be a special semimartingale as well, meaning that $\mathcal{L}(Z)$ is a special semimartingale up to infinity. Furthermore, Itô’s formula applied to the inverse function $(0, \infty) \ni x \mapsto x^{-1}$ implies that \tilde{Z}_-^{-1} is a semimartingale up to infinity and, since \tilde{Z}_- is locally bounded, $\mathcal{L}(\tilde{Z}_-^{-1}) = \tilde{Z}_- \cdot \tilde{Z}_-^{-1}$ is a semimartingale, which finishes the proof. □

Remark 10.3 In this paper we consider “semimartingales up to time T ” and “stochastic integration up to time T ” where T is a stopping time rather than “semimartingales up to infinity” and “stochastic integration up to infinity”. One can use all the results of this section applying them to the processes stopped at time T —divergence from the usual notion of integrability appears only when $\mathbb{P}[T = \infty] > 0$.

Appendix 3 σ -Localization

A good account of the concept of σ -localization is given in [23]. Here we recall briefly what is needed for our purposes. For a semimartingale Z and a predictable set Σ , define $Z^\Sigma := \mathbb{I}_\Sigma \cdot Z$.

Definition 11.1 Let \mathcal{Z} be a class of semimartingales. Then the corresponding σ -localized class \mathcal{Z}_σ is defined as the set of all semimartingales Z for which there exists an increasing sequence $(\Sigma_n)_{n \in \mathbb{N}}$ of predictable sets such that $\Sigma_n \uparrow \Omega \times \mathbb{R}_+$ (up to evanescence) and $Z^{\Sigma_n} \in \mathcal{Z}$ for all $n \in \mathbb{N}$.

When the corresponding class \mathcal{Z} has a name (like “supermartingales”) we baptize the class \mathcal{Z}_σ with the same name preceded by “ σ -” (like “ σ -supermartingales”).

The concept of σ -localization is a natural extension of the well-known concept of localization along a sequence $(\tau_n)_{n \in \mathbb{N}}$ of stopping times, as one can easily see by considering the predictable sets $\Sigma_n \equiv \llbracket 0, \tau_n \rrbracket := \{(\omega, t) \in \Omega \times \mathbb{R}_+ \mid 0 \leq t \leq \tau_n(\omega)\}$.

Let us define the set \mathcal{U} of semimartingales Z such that the collection of random variables $\{Z_\tau \mid \tau \text{ is a stopping time}\}$ is uniformly integrable—also known in the literature as semimartingales of class (D). The elements of \mathcal{U} admit the *Doob–Meyer decomposition* $Z = A + M$ into a predictable finite variation part A with $A_0 = 0$ and $\mathbb{E}[\text{Var}(A)_\infty] < \infty$ and a uniformly integrable martingale M . It is then obvious that the localized class \mathcal{U}_{loc} corresponds to all special semimartingales; they are exactly the ones which admit a Doob–Meyer decomposition as before, but where now A is only a predictable, finite variation process with $A_0 = 0$ and M a local martingale. Let us remark that the local supermartingales (resp., local submartingales) correspond to these elements of \mathcal{U}_{loc} with A decreasing (resp., increasing). This last result can be found, for example, in [20], Proposition 2.18, and the discussion following it.

One can have very intuitive interpretations of some σ -localized classes in terms of the predictable characteristics of Z .

Proposition 11.2 Consider a scalar semimartingale Z and let (b, c, ν) be the triplet of predictable characteristics of Z relative to the canonical truncation function and the operational clock G . Then

- (1) Z belongs to \mathcal{U}_{loc} if and only if the predictable process $\int |x| \mathbb{I}_{\{|x|>1\}} \nu(dx)$ is G -integrable.
- (2) Z belongs to \mathcal{U}_σ if and only if $\int |x| \mathbb{I}_{\{|x|>1\}} \nu(dx) < \infty$.
- (3) Z is a σ -supermartingale if and only if $\int |x| \mathbb{I}_{\{|x|>1\}} \nu(dx) < +\infty$ and $b + \int x \mathbb{I}_{\{|x|>1\}} \nu(dx) \leq 0$.

Proof The first statement follows from the fact that a 1-dimensional semimartingale Z is a special semimartingale (i.e., a member of \mathcal{U}_{loc}) if and only if $[|x| \mathbb{I}_{\{|x|>1\}}] * \hat{\eta}$ is a finite, increasing predictable process (one can consult [21], Proposition 2.29, for this fact). The second statement follows easily from the first and σ -localization. Finally, the third follows from the fact that for a process in \mathcal{U}_{loc} the predictable finite variation part is given by the process $(b + \int [x \mathbb{I}_{\{|x|>1\}}] \nu(dx)) \cdot G$, using the last remark before the proposition, the first part of the proposition, and σ -localization. □

Results like the last proposition are very intuitive, because $b + \int x \mathbb{1}_{\{|x|>1\}} \nu(dx)$ represents the infinitesimal drift rate of the semimartingale Z ; we expect this rate to be negative (resp., positive) in the case of σ -supermartingales (resp., σ -submartingales). The importance of σ -localization is that it allows us to talk directly about drift rates of processes, rather than about drifts. Sometimes drift rates exist, but cannot be integrated to give a drift process; this is when the usual localization technique fails, and the concept of σ -localization becomes useful.

The following result gives sufficient conditions for a σ -supermartingale to be a local supermartingale (or even plain supermartingale).

Proposition 11.3 *Suppose that Z is a scalar semimartingale with triplet of predictable characteristics (b, c, ν) .*

- (1) *Suppose that Z is a σ -supermartingale. Then the following are equivalent:*
 - (a) *Z is a local supermartingale.*
 - (b) *The positive, predictable process $\int(-x)\mathbb{1}_{\{x<-1\}}\nu(dx)$ is G -integrable.*
- (2) *If Z is a σ -supermartingale (resp., σ -martingale) and bounded from below by a constant then it is a local supermartingale (resp., local martingale). If, furthermore, $\mathbb{E}[Z_0^+] < \infty$, it is a supermartingale.*
- (3) *If Z is bounded from below by a constant then it is a supermartingale if and only if $\mathbb{E}[Z_0^+] < \infty$ and $b + \int x \mathbb{1}_{\{|x|>1\}} \nu(dx) \leq 0$.*

Proof For the proof of (1), the implication (a) \Rightarrow (b) follows from part (1) of Proposition 11.2. For (b) \Rightarrow (a), assume that $\int(-x)\mathbb{1}_{\{x<-1\}}\nu(dx)$ is G -integrable. Since Z is a σ -supermartingale, it follows from part (3) of Proposition 11.2 that $\int x \mathbb{1}_{\{x>1\}} \nu(dx) \leq -b + \int(-x)\mathbb{1}_{\{x<-1\}}\nu(dx)$. Now this last inequality implies that $\int |x| \mathbb{1}_{\{|x|>1\}} \nu(dx) \leq -b + 2 \int(-x)\mathbb{1}_{\{x<-1\}}\nu(dx)$; the last dominating process is G -integrable, thus $Z \in \mathcal{U}_{loc}$ (again, part (1) of Proposition 11.2). The special semimartingale Z has predictable finite variation part equal to $(b + \int x \mathbb{1}_{\{x>1\}} \nu(dx)) \cdot G$, which is decreasing, so that Z is a local supermartingale.

For part (2), we can of course assume that Z is positive. We discuss the case of a σ -supermartingale; the σ -martingale case follows in the same way. According to part (1) of this proposition, we only need to show that $\int(-x)\mathbb{1}_{\{x<-1\}}\nu(dx)$ is G -integrable. But since the negative jumps of Z are bounded in magnitude by Z_- , we have that $\int(-x)\mathbb{1}_{\{x<-1\}}\nu(dx) \leq (Z_-)\nu[x < -1]$, which is G -integrable, because $\nu[x < -1]$ is G -integrable and Z_- is locally bounded. Now, if we further assume that $\mathbb{E}[Z_0] < \infty$, Fatou’s lemma for conditional expectations gives us that the positive local supermartingale Z is a supermartingale.

Let us move on to part (3) and assume that Z is positive. First assume that Z is a supermartingale. Then, of course, we have $\mathbb{E}[Z_0] < \infty$ and that Z is an element of \mathcal{U}_σ (and even of \mathcal{U}_{loc}) and part (3) of Proposition 11.2 ensures that $b + \int x \mathbb{1}_{\{|x|>1\}} \nu(dx) \leq 0$. Now assume that Z is a positive semimartingale with $\mathbb{E}[Z_0] < \infty$ and that $b + \int x \mathbb{1}_{\{|x|>1\}} \nu(dx) \leq 0$. Then, of course, we have that $\int x \mathbb{1}_{\{x>1\}} \nu(dx) < \infty$. Also, since Z is positive, we always have that $\nu[x < -Z_-] = 0$ so that $\int(-x)\mathbb{1}_{\{x<-1\}}\nu(dx) < \infty$, too. Part (2) of Proposition 11.2 will give us that $Z \in \mathcal{U}_\sigma$, and part (3) of the same proposition that Z is a σ -supermartingale. Finally, part (2) of this proposition gives us that Z is a supermartingale. \square

The special case of result (3) of Proposition 11.3 when Z is a σ -martingale is sometimes called “the Ansel–Stricker theorem,” since it first appeared (in a slightly different, but equivalent form) in [3]. In [23] one can find the proof of the case when Z is a σ -supermartingale bounded from below with $\mathbb{E}[Z_0^+] < \infty$.

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