Finance Stoch (2014) 18:75–114
DOI 10.1007/s00780-013-0216-5

Abstract, classic, and explicit turnpikes

**Paolo Guasoni · Constantinos Kardaras ·
Scott Robertson · Hao Xing**

Received: 8 February 2012 / Accepted: 26 April 2013 / Published online: 13 November 2013

Abstract Portfolio turnpikes state that as the investment horizon increases, optimal portfolios for generic utilities converge to those of isoelastic utilities. This paper proves three kinds of turnpikes. In a general semimartingale setting, the *abstract* turnpike states that optimal final payoffs and portfolios converge under their myopic probabilities. In diffusion models with several assets and a single state variable, the *classic* turnpike demonstrates that optimal portfolios converge under the physical probability. In the same setting, the *explicit turnpike* identifies the limit of finite-horizon optimal portfolios as a long-run myopic portfolio defined in terms of the solution of an ergodic HJB equation.

Keywords Portfolio choice · Incomplete markets · Long-run · Utility functions · Turnpikes

P. Guasoni
School of Mathematical Sciences, Dublin City University, Glasnevin, Dublin 9, Ireland

P. Guasoni
Department of Mathematics and Statistics, Boston University, 111 Cummington St, Boston, MA 02215, USA
e-mail: guasoni@bu.edu

C. Kardaras · H. Xing (✉)
Department of Statistics, London School of Economics and Political Science, 10 Houghton St, London, WC2A 2AE, UK
e-mail: h.xing@lse.ac.uk

C. Kardaras
e-mail: k.kardaras@lse.ac.uk

S. Robertson
Department of Mathematical Sciences, Carnegie Mellon University, Wean Hall 6113, Pittsburgh, PA 15213, USA
e-mail: scottrob@andrew.cmu.edu

Mathematics Subject Classification 93E20 · 60H30

JEL Classification G11 · C61

1 Introduction

Explicit formulas are rare in portfolio choice, and arise mainly with isoelastic utilities and long horizons. As a result, little is known about general optimal portfolios, especially if investment opportunities are stochastic and markets are incomplete. Turnpike theorems provide tractable and approximately optimal portfolios for a large class of utility functions, and long but finite-horizons. Informally, these results state the following: when the investment horizon is distant, the optimal portfolio for a generic utility approaches that of an isoelastic, or power, utility, where the power is determined by the marginal growth rate of the original utility function. Unfortunately, available results focus on either independent returns, or on complete markets.

This paper proves turnpike theorems in a general framework, which includes discrete and continuous time, and nests diffusion models with several assets, stochastic drifts, volatilities, and interest rates. The paper departs from the existing literature, in which either asset returns are independent over time, or markets are complete. It is precisely when both these assumptions fail that portfolio choice becomes most challenging, and turnpike theorems are most useful.

Our results have three broad implications. First, turnpike theorems are a powerful tool in portfolio choice, because they apply not only when optimal portfolios are myopic but also when the *intertemporal hedging* component is present. Finding this component is the central problem of portfolio choice, and the only tractable but non-trivial analysis is based on isoelastic utilities, combined with long horizon asymptotics. Turnpike theorems make this analysis relevant for a large class of utility functions, and for long, but finite, horizons.

Second, we clarify the roles of preferences and market structure for turnpike results. Under regularity conditions on utility functions, an *abstract* version of a turnpike theorem holds regardless of market structure, as long as utility maximization is well posed, and longer horizons lead to higher payoffs. This abstract turnpike yields the convergence of optimal portfolios to their isoelastic limit under *myopic* probabilities $\mathbb{P}^T$, which change with the horizon T. Market structure becomes crucial to pass from the abstract to the *classic turnpike* theorem, in which convergence holds under the physical probability $\mathbb{P}$.

Third, we prove a new kind of result, the *explicit turnpike*, in which the limit portfolio is identified as the *long-run optimal* portfolio, the latter being a stationary portfolio identified by an ergodic Hamilton–Jacobi–Bellman (HJB) equation. This result provides a tractable and asymptotically optimal portfolio for the long-term investment with generic utilities. Moreover, it offers the first theoretical basis for the long-standing practice of interpreting solutions of ergodic HJB equations as long-run limits of utility maximization problems.[1] We show that this intuition is indeed cor-

[1] This interpretation underpins the literature on *risk-sensitive control*, introduced by Fleming and McEneaney [15], and applied to optimal portfolio choice by Bielecki et al. [4], Bielecki and Pliska [3], Fleming and Sheu [16, 17], Nagai and Peng [41, 42], among others.

rect for a large class of diffusion models, and that its scope includes a broader class of utility functions.

Portfolio turnpikes start with the work of Mossin [40] on affine risk tolerance, i.e., $-U'(x)/U''(x) = ax + b$, which envisions many of the later developments. In his concluding remarks, he writes: *"Do any of these results carry over to arbitrary utility functions? They seem reasonable enough, but the generalization does not appear easy to make. As one usually characterizes those problems one hasn't been able to solve oneself: this is a promising area for future research."*

Leland [39] coins the expression *portfolio turnpike*, extending Mossin's result to larger classes of utilities, followed by Ross [48] and Hakansson [23]. Huberman and Ross [26] prove a necessary and sufficient condition for the turnpike property. As in the previous literature, they consider discrete-time models with independent returns. Cox and Huang [7] prove the first turnpike theorem in continuous time using contingent claim methods. Jin [28] extends their results to include consumption, and Huang and Zariphopoulou [25] obtain similar results using viscosity solutions. Dybvig et al. [12] dispose of the assumption of independent returns, proving a turnpike theorem for complete markets in the Brownian filtration, while Detemple and Rindisbacher [11] obtain a portfolio decomposition formula for complete markets, which allows to compute turnpike portfolios in certain models.

In summary, the literature either exploits independent returns, which make dynamic programming attractive, or complete markets, which make martingale methods convenient. Since market completeness and independence of returns have a tenuous relation, neither of these concepts appears to be central to turnpike theorems. Indeed, in this paper both assumptions are dropped.

The basic intuition of portfolio turnpikes is that if wealth grows indefinitely, then investment policies should depend only on the behavior of the utility function at high levels of wealth, and if two utility functions are close, so should be their optimal portfolios. The question is whether the utilities themselves, or marginal utilities, or risk aversions should be close for portfolios to converge, and this paper provides precise conditions that are valid even in incomplete markets. Portfolio turnpikes can also be seen as stability results for optimal investment problems with respect to the horizon, and stability typically involves some rather delicate conditions (cf. Larsen and Žitković [38] and Cheridito and Summer [5]).

The main results are in Sect. 2, which is divided into three parts. The first part shows the conditions leading to the abstract turnpike, whereby optimal final payoffs and portfolios converge under the myopic probabilities. Assumption 2.1 requires a marginal utility that is asymptotically isoelastic as wealth increases, and a well-posed utility maximization problem. The abstract turnpike is a crucial step towards stronger turnpike theorems, because it reduces the comparison of the optimal portfolio for a generic utility and its isoelastic counterpart to the comparison of the optimal isoelastic finite-horizon portfolio with its long-run limit.

The second part of Sect. 2 introduces a class of diffusion models with several assets but with a single state variable driving expected returns, volatilities and interest rates. The discussion starts with a heuristic argument, which shows the relation between finite-horizon and long-run portfolio choice for isoelastic utilities. A rigorous account of classic and explicit turnpike theorems for the diffusion models follows in the last part of Sect. 2. For an ergodic state variable, a classic turnpike theorem holds: optimal portfolios of generic utility functions converge to their isoelastic counterparts. The same machinery leads to the explicit turnpike, in which optimal finite-horizon portfolios for a generic utility converge to the long-run optimal portfolio, defined via the solution of an ergodic HJB equation. Section 2 concludes with an application to target-date retirement funds, which shows that a fund manager who tries to maximize the weighted welfare of participants—like a social planner—tends to act on behalf of the least risk-averse investors.

Section 3 contains the proofs of the abstract turnpike, while the classic and the explicit turnpike for diffusions are proved in Sect. 4. The first part of Sect. 3 proves the convergence of the ratio of final payoffs, while the second part derives the convergence of wealth processes. Section 4 studies the properties of the long-run measure and the value function, and continues with the convergence of densities and wealth processes, from which the classic and explicit turnpikes follow.

In conclusion, this paper shows that turnpike theorems are a useful tool to make portfolio choice tractable, even in the most intractable setting of incomplete markets combined with stochastic investment opportunities. Still, these results are likely to admit extensions to more general settings, like diffusions with multiple state variables. As gracefully put by Mossin, this is a promising area for future research.

2 Main results

This section contains the statements of the main results and their implications. The first subsection states an abstract version of the turnpike theorem, which focuses on payoff spaces and wealth processes, in a general semimartingale model. In this setting, asymptotic conditions on the utility functions and on wealth growth imply that as the horizon increases, optimal wealths and optimal portfolios converge to their isoelastic counterparts.

The defining feature of the abstract turnpike is that convergence takes place under a family of probability measures $(\mathbb{P}^T)_{T\geq 0}$ that change with the horizon. By contrast, in the *classic* turnpike, which is the desired result, the convergence holds under the physical probability measure $\mathbb{P}$. While the abstract turnpike provides an important preliminary step towards the classic turnpike, passing from the abstract to the classic turnpike theorem requires convergence of the probabilities $\mathbb{P}^T$, which in turn commands additional assumptions. The second and third subsections achieve this task for a class of diffusion models with several risky assets, and with a single state variable driving investment opportunities. This class nests several models in the literature, and allows return predictability, stochastic volatility, and stochastic interest rates.

The *explicit* turnpike—stated at the end of the third subsection—holds for the same class of diffusion models. Whereas in the classic turnpike the benchmark is the optimal portfolio for isoelastic utility with the same finite-horizon, in the explicit turnpike the benchmark is the long-run optimal portfolio; that is, the portfolio which achieves the maximal expected utility growth rate for an isoelastic investor. In contrast to the finite-horizon optimal portfolio, the long-run optimal portfolio is independent of the investment horizon, and is more tractable than its finite-horizon counterpart. In the explicit turnpike, as the investment horizon increases, the finite-horizon portfolios for a generic utility converge to the corresponding isoelastic long-run optimal portfolio, providing tractable and asymptotically optimal portfolios for long-term investment with generic utilities.

2.1 The abstract turnpike theorem

Consider two investors, one with constant relative risk aversion (henceforth CRRA) equal to $1-p$ (i.e., power utility x^p/p for $0 \neq p < 1$ or logarithmic utility $\log x$ for $p=0$), the other with a generic utility function $U:\mathbb{R}_+ \to \mathbb{R}$. The marginal utility ratio $\mathfrak{R}(x)$ measures how close U is to the reference utility; it is defined by

$$\mathfrak{R}(x) := \frac{U'(x)}{x^{p-1}}, \quad x > 0. \tag{2.1}$$

Assumption 2.1 The utility function $U:\mathbb{R}_+ \to \mathbb{R}$ is continuously differentiable, strictly increasing, strictly concave, and satisfies the Inada conditions $U'(0)=\infty$ and $U'(\infty)=0$. The marginal utility ratio satisfies

$$\lim_{x\to\infty} \mathfrak{R}(x) = 1. \tag{2.2}$$

Condition (2.2) means that investors have similar marginal utilities when wealth is high, and is the basic assumption on *preferences* for turnpike theorems; see Dybvig et al. [12], Huang and Zariphopoulou [25].

Both investors trade in a frictionless market with one safe and d risky assets. Let $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathcal{F}, \mathbb{P})$ be a filtered probability space with $(\mathcal{F}_t)_{t\geq 0}$ a right-continuous filtration. It is important for the developments in the paper not to include all negligible sets in $\mathcal{F}$ into $\mathcal{F}_0$. However, to align with references on finite-horizon problems, where only negligible sets in $\mathcal{F}_T$, for some $T\geq 0$, are included, we include all N-negligible sets into $\mathcal{F}_0$.[2] The safe asset, denoted by $(S^0_t)_{t\geq 0}$, and the risky assets $(S^i_t)^{i=1,\dots,d}_{t\geq 0}$ satisfy.

Assumption 2.2 S^0 has RCLL (right-continuous with left limits) paths, and there exist two deterministic functions $\underline{S}^0, \overline{S}^0$ on $(0,\infty)$ such that $0 < \underline{S}^0_t \leq S^0_t \leq \overline{S}^0_t$ for all $t>0$ and

$$\lim_{T\to\infty} \underline{S}^0_T = \infty. \tag{2.3}$$

This condition means that growth continues over time, and is the main *market* assumption in the turnpike literature. It implies that the riskless discount factor declines to zero in the long run. Denote the discounted prices of risky assets by $\widetilde{S}^i = S^i/S^0$

[2] A subset A of Ω is N-negligible if there exists a sequence $(B_n)_{n\geq 0}$ of subsets of Ω such that for all $n\geq 0$, we have $B_n \in \mathcal{F}_n$ and $\mathbb{P}[B_n]=0$, and $A \subset \bigcup_{n\geq 0} B_n$. This notion is introduced in [2, Definition 1.3.23] and [43]. Such a completion of $\mathcal{F}_0$ ensures, for all $T\geq 0$, that the space $(\Omega, \mathcal{F}_T, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ satisfies the usual conditions. Hence all references below on finite-horizon problems with completed filtration can be used in this paper.

for $i = 1, \dots, d$, and set $\widetilde{S} = (\widetilde{S}^i_t)^{i=1,\dots,d}_{t\geq 0}$. The following assumption is equivalent to the absence of arbitrage, in the sense of no free lunch with vanishing risk [9, 10]. In particular, $S = (S^i_t)^{i=1,\dots,d}_{t\geq 0}$ is an $\mathbb{R}^d$-valued semimartingale with RCLL paths.

Assumption 2.3 For all $T \in \mathbb{R}_+$, there exists a probability $\mathbb{Q}^T$ that is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$ and such that $\widetilde{S}$ is a (vector) sigma-martingale on $[0, T]$.

Starting from unit initial capital, each investor trades with some admissible strategy H; this is an S-integrable and $(\mathcal{F}_t)$-predictable $\mathbb{R}^d$-valued process such that $\widetilde{X}^H_t := 1 + \int_0^t H_u \, d\widetilde{S}_u \geq 0$ $\mathbb{P}$-a.s. for all $t \geq 0$. We denote wealth processes by $X^H = S^0 \widetilde{X}^H$, and their class by $\mathcal{X} := \{X^H : H \text{ is admissible}\}$.

Both investors seek to maximize the expected utility of their terminal wealth at some time horizon T. Using the index 0 for the CRRA investor and 1 for the generic investor, their optimization problems are

$$u^{0,T} = \sup_{X\in\mathcal{X}} \mathbb{E}^{\mathbb{P}}[X_T^p/p], \qquad u^{1,T} = \sup_{X\in\mathcal{X}} \mathbb{E}^{\mathbb{P}}[U(X_T)].$$

When $p = 0$, $u^{0,T}$ is understood as $\sup_{X\in\mathcal{X}} \mathbb{E}^{\mathbb{P}}[\log(X_T)]$. The next assumption requires that these problems are well posed. It holds under the simple criteria in Karatzas and Žitković [33, Remark 8].

Assumption 2.4 For $0 \leq p < 1$, $u^{0,T} < \infty$ for all $T > 0$.

The previous assumption and (2.2) together imply that we have $u^{1,T} < \infty$ when $0 \leq p < 1$ and $T > 0$.[3] Moreover, $u^{i,T} \leq 0$ for $i = 0, 1$ when $p < 0$ and $T > 0$. Therefore the utility maximization problems for both investors are well posed for all horizons. It then follows from [33] that under Assumptions 2.1–2.4, the optimal wealth processes $X^{i,T}$ exist for $i = 0, 1$ and any $T \geq 0$. In addition, $u^{i,T} > -\infty$, because both investors can invest all their wealth in S^0 alone, and S^0_T is bounded away from zero by a constant.

The central objects in the turnpike theorem are the ratio of the optimal wealth processes and their stochastic logarithm,

$$r^T_u := \frac{X^{1,T}_u}{X^{0,T}_u}, \qquad \Pi^T_u := \int_0^u \frac{dr^T_v}{r^T_{v-}}, \qquad \text{for } u \in [0, T]. \tag{2.4}$$

They are well defined by Remark 3.2 below. Moreover, $r^T_0 = 1$ since both investors start with the same initial capital. To state the abstract turnpike result, define the

[3] For any $\epsilon > 0$, there exists M_ϵ such that $U'(x) \leq (1+\epsilon)x^{p-1}$ for $x \geq M_\epsilon$. Integrating the previous inequality on (M_ϵ, x) yields $U(x) \leq (1+\epsilon)(x^p - M_\varepsilon^p)/p + U(M_\varepsilon)$, when $x \geq M_\epsilon$ and $0 < p < 1$, from which the claim follows. The proof for the case $p = 0$ is similar.

myopic probabilities[4] $(\mathbb{P}^T)_{T\geq 0}$ by

$$\frac{d\mathbb{P}^T}{d\mathbb{P}} := \frac{(X_T^{0,T\,p})}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,T\,p})]}. \tag{2.5}$$

The above densities are well defined and strictly positive (cf. Assumption 2.4 and Remark 3.2); hence $\mathbb{P}^T$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_T$. Moreover, $\mathbb{P}^T = \mathbb{P}$ in the logarithmic case $p = 0$. The expression *myopic probability* is used since an investor with relative risk aversion $1 - p$ under the probability $\mathbb{P}$ selects the same optimal payoff as an investor with logarithmic utility under the probability $\mathbb{P}^T$.

With the above definitions, the abstract version of the turnpike theorem reads as follows.

Proposition 2.5 (Abstract turnpike) *Let Assumptions 2.1–2.4 hold. Then, for any* $\epsilon > 0$:

(a) $\lim_{T\to\infty} \mathbb{P}^T(\sup_{u\in[0,T]} |r_u^T - 1| \geq \epsilon) = 0$;
(b) $\lim_{T\to\infty} \mathbb{P}^T([\varPi^T, \varPi^T]_T \geq \epsilon) = 0$, *where* $[\cdot,\cdot]$ *denotes the square bracket of semimartingales.*

Since $\mathbb{P}^T \equiv \mathbb{P}$ for $p = 0$, convergence holds under $\mathbb{P}$ in the case of logarithmic utility. In this case, Proposition 2.5 is already the classic turnpike. Moreover, since the optimal portfolio for logarithmic utility is myopic [20, Proposition 2.1] is then also the explicit turnpike.

To gain intuition into the structure of $[\varPi^T, \varPi^T]$, consider an Itô process market with the discounted asset price dynamics given by

$$\frac{d\widetilde{S}_u^j}{\widetilde{S}_u^j} = \mu_u^j\,du + \sum_{k=1}^n \sigma_u^{jk}\,dW_u^k, \quad j = 1, \ldots, d,$$

where μ and σ are, respectively, $\mathbb{R}^d$- and $\mathbb{R}^{d\times n}$-valued predictable processes and $W = (W^1, \ldots, W^n)'$, with $'$ representing transposition, is an $\mathbb{R}^n$-valued Brownian motion. Here, the discounted optimal wealth processes satisfy

$$d\widetilde{X}_u^{i,T} = \widetilde{X}_u^{i,T}\big(\pi_u^{i,T}\big)'(\mu_u\,du + \sigma_u\,dW_u), \quad i = 0, 1,$$

where $(\pi^{j,T})_{u\geq 0}^{j=1,\ldots,d}$ represents the proportions of wealth invested in each risky asset. In this case, $[\varPi^T, \varPi^T]$ measures the squared distance between the portfolios $\pi^{1,T}$ and $\pi^{0,T}$, weighted by $\varSigma = \sigma\sigma'$, because

$$\big[\varPi^T, \varPi^T\big]_\cdot = \int_0^\cdot \big(\pi_u^{1,T} - \pi_u^{0,T}\big)'\varSigma_u\big(\pi_u^{1,T} - \pi_u^{0,T}\big)\,du.$$

[4] These probabilities already appear in the work of Kramkov and Sîrbu [35–37] under the name of **R**.

2.1.1 From abstract to classic turnpikes

Except for logarithmic utility, Proposition 2.5 is not a classic turnpike theorem, in that convergence holds under the probability measures $(\mathbb{P}^T)_{T\geq 0}$ which change with T. However, the convergence in Proposition 2.5 holds on the entire (growing) time interval $[0,T]$. By contrast, in the classic turnpike, convergence holds under the physical measure $\mathbb{P}$ for any (fixed) horizon $[0,t]$, where $t>0$.

To pass from the abstract to the classic turnpike, observe two facts. First, since $\mathbb{P}^T$ is constructed by solving the finite-horizon problem for an isoelastic investor, the role of the abstract turnpike is to reduce the problem of comparing optimal portfolios between generic and isoelastic utilities to comparing optimal portfolios between finite-horizon and long-run isoelastic investors. Second, Proposition 2.5 implies that for any $t>0$, both optimal wealth processes and portfolios are close, under $\mathbb{P}^T$, in the time window $[0,t]$. Indeed, for any $\epsilon>0$, we have

$$\lim_{T\to\infty}\mathbb{P}^T\Big(\sup_{u\in[0,t]}\big|r_u^T-1\big|\geq\epsilon\Big)=0 \quad\text{and}\quad \lim_{T\to\infty}\mathbb{P}^T\big(\big[\Pi^T,\Pi^T\big]_t\geq\epsilon\big)=0.$$

Since the events $\{\sup_{u\in[0,t]}|r_u^T-1|\geq\epsilon\}$ and $\{[\Pi^T,\Pi^T]_t\geq\epsilon\}$ are $\mathcal{F}_t$-measurable, the classic turnpike will follow precisely when the measure $\mathbb{P}$ is *contiguous* with respect to $(\mathbb{P}^T)_{T\geq t}$ on $\mathcal{F}_t$ for all $t>0$ (see [27, 29] for the contiguity of measures). The following lemma, used in the sequel, connects the abstract and classic turnpikes.

Lemma 2.6 *Let $\mathbb{Q}$, $\widetilde{\mathbb{Q}}$, and $(\mathbb{Q}^T)_{T\geq t}$ be measures on $(\Omega,\mathcal{F}_t)$ such that $\mathbb{Q}\sim\widetilde{\mathbb{Q}}$. Let $(A_T)_{T\geq t}\subset\mathcal{F}_t$ be such that $\lim_{T\to\infty}\mathbb{Q}^T[A_T]=0$. If $\mathbb{Q}^T\ll\widetilde{\mathbb{Q}}$ on $\mathcal{F}_t$ for each $T\geq t$, and if there exists an $\varepsilon>0$ such that $\lim_{T\to\infty}\widetilde{\mathbb{Q}}[d\mathbb{Q}^T/d\widetilde{\mathbb{Q}}\geq\varepsilon]=1$ (in particular, if*[5] *$\widetilde{\mathbb{Q}}$-$\lim_{T\to\infty}d\mathbb{Q}^T/d\widetilde{\mathbb{Q}}=1$), then $\lim_{T\to\infty}\mathbb{Q}[A_T]=0$.*

2.2 A turnpike for myopic strategies with independent returns

The density between $\mathbb{P}^T$ and $\mathbb{P}$ on $\mathcal{F}_t$ is given by the projection

$$\left.\frac{d\mathbb{P}^T}{d\mathbb{P}}\right|_{\mathcal{F}_t}=\frac{\mathbb{E}_t^{\mathbb{P}}[(X_T^{0,T})^p]}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,T})^p]}. \tag{2.6}$$

In fact, the densities in (2.6) become constant in T when the finite-horizon optimal CRRA strategy is myopic (i.e., does not depend upon T) and is such that its wealth process has independent returns. Under these assumptions, which are ubiquitous within the literature (see Dybvig et al. [12, Theorem 1] for a notable exception), the classic turnpike theorem follows:

Corollary 2.7 (i.i.d. myopic turnpike) *If, in addition to Assumptions 2.1–2.4,*

1. $X_t^{0,T}=X_t^{0,S}$ *$\mathbb{P}$-a.s. for all $t\leq S,T$ (myopic optimality);*

[5] The notation $\widetilde{\mathbb{Q}}$-$\lim_{T\to\infty}$ is short for the limit in probability under $\widetilde{\mathbb{Q}}$.

2. $X_s^{0,T}/X_t^{0,T}$ *and* $\mathcal{F}_t$ *are independent under* $\mathbb{P}$ *for all* $t \le s \le T$ (*independent returns*),

then, for any $\epsilon, t > 0$, *we have*

(a) $\lim_{T\to\infty} \mathbb{P}(\sup_{u\in[0,t]} |r_u^T - 1| \ge \epsilon) = 0$;
(b) $\lim_{T\to\infty} \mathbb{P}([\Pi^T, \Pi^T]_t \ge \epsilon]) = 0$.

Typically, if asset prices have independent returns, the optimal strategy for a CRRA investor generates a myopic portfolio with independent returns. This is the case, for example, if asset prices follow exponential Lévy processes, as in Goll and Kallsen [31]. However, as Example 2.22 below shows, a myopic CRRA portfolio alone is not sufficient to ensure that $\mathbb{P}^T$ is independent of T.

The assumptions of Corollary 2.7 exclude those models in which portfolio choice is least tractable, and turnpike results are needed the most. The next section proves classic and explicit turnpikes for diffusion models in which returns need not be independent, and the market may be incomplete.

2.3 A turnpike for diffusions

This section introduces a class of diffusion models in which a single state variable drives investment opportunities. Conditions are given under which both the classic and explicit turnpikes hold. For the purposes of this whole subsection, we assume that $p \neq 0$; the case $p = 0$ is simpler and has already been studied in much greater generality in the previous subsections.

2.3.1 The model

The state variable takes values in an interval $E = (\alpha, \beta)$, with $-\infty \le \alpha < \beta \le \infty$, and has dynamics

$$dY_t = b(Y_t)\,dt + a(Y_t)\,dW_t. \tag{2.7}$$

The market includes a safe rate $r(Y_t)$ and d risky assets, with discounted prices $\tilde{S}^i$ satisfying $d\tilde{S}_t^i/\tilde{S}_t^i = dR_t^i$, $i = 1, \dots, d$, where the cumulative excess return process R has dynamics

$$dR_t^i = \mu_i(Y_t)\,dt + \sum_{j=1}^{d} \sigma_{ij}(Y_t)\,dZ_t^j, \quad i = 1, \dots, d. \tag{2.8}$$

W and Z are, respectively, 1- and d-dimensional Brownian motions with constant correlation ρ, i.e., $\rho = (\rho^1, \dots, \rho^d)' \in \mathbb{R}^d$ with $d\langle Z^i, W\rangle_t = \rho^i\,dt$ for $i = 1, \dots, d$. Hence one can write $Z = \rho W + \overline{\rho} B$ with a d-dimensional Brownian motion B independent of W and $\overline{\rho} \in \mathbb{R}^{d\times d}$ being a square root of $1_{n\times n} - \rho\rho'$, where $1_{n\times n}$ denotes the $n \times n$ identity matrix. Set

$$\Sigma := \sigma\sigma', \qquad A := a^2, \qquad \Upsilon := \sigma\rho a.$$

The first two assumptions concern the well-posedness of the model given in (2.7) and (2.8). Recall that for $\gamma \in (0, 1]$ and an integer k, a function $f : E \to \mathbb{R}$ is *locally* $C^{k,\gamma}$ *on* E if for all bounded, open, connected $D \subset E$ such that $\bar{D} \subset E$, f is in the Hölder space $C^{k,\gamma}(\bar{D})$ (Evans [13, Chap. 5.1]). For integers n, m, $C^{k,\gamma}(E, \mathbb{R}^{n\times m})$ is the set of all $n \times m$ matrix-valued f for which each component f_{ij} is locally $C^{k,\gamma}$ on E. With $\mathbb{R}^n \equiv \mathbb{R}^{n\times 1}$, we impose

Assumption 2.8 We have $r \in C^{\gamma}(E, \mathbb{R})$, $b \in C^{1,\gamma}(E, \mathbb{R})$, $\mu \in C^{1,\gamma}(E, \mathbb{R}^d)$, $A \in C^{2,\gamma}(E, \mathbb{R})$, $\Sigma \in C^{2,\gamma}(E, \mathbb{R}^{d\times d})$, and $\Upsilon \in C^{2,\gamma}(E, \mathbb{R}^d)$ for some $\gamma \in (0, 1]$. For all $y \in E$, Σ is strictly positive definite and A is strictly positive.

These regularity conditions imply the local existence and uniqueness of a solution (R, Y). The next assumption ensures the existence of a unique global solution, by requiring that Feller's test for explosions is negative; see [46, Theorem 5.1.5].

Assumption 2.9 There is some $y_0 \in E$ such that

$$\int_{\alpha}^{y_0} \frac{1}{A(y)m(y)} \left(\int_{y}^{y_0} m(z)\, dz \right) dy = \infty = \int_{y_0}^{\beta} \frac{1}{A(y)m(y)} \left(\int_{y_0}^{y} m(z)\, dz \right) dy,$$

where the speed measure density is defined as

$$m(y) := \big(A(y)\big)^{-1} \exp\left(\int_{y_0}^{y} 2b(z)/A(z)\, dz \right).$$

Assumption 2.9 implies that the model for (R, Y) is well posed in that it admits a solution. This statement is made precise within the setting of martingale problems. For a fixed integer n, let Ω be the space of continuous maps $\omega : \mathbb{R}_+ \to \mathbb{R}^n$ and $\mathcal{B} = (\mathcal{B}_t)_{t\geq 0}$ the filtration generated by the coordinate process Ξ defined by $\Xi_t(\omega) = \omega_t$ for $\omega \in \Omega$. Let $\mathcal{F} = \sigma(\Xi_t, t \geq 0)$ and $\mathcal{F}_t = \mathcal{B}_{t+} \vee \{N\text{-negligible sets}\}$; cf. footnote 2. For an open, connected set $D \subset \mathbb{R}^n$ and $\gamma \in (0, 1]$, let $\check{A} \in C^{2,\gamma}(D, \mathbb{R}^{n\times n})$ be pointwise positive definite and let $\check{b} \in C^{1,\gamma}(D, \mathbb{R}^n)$. Define the second-order elliptic operator $\check{L}$ by

$$\check{L} := \frac{1}{2} \sum_{i,j=1}^{n} \check{A}_{ij} \frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \check{b}_i \frac{\partial}{\partial x_i}. \tag{2.9}$$

Definition 2.10 A family of probability measures $(\mathbb{P}^x)_{x\in D}$ on $(\Omega, \mathcal{F})$ is a *solution to the martingale problem for* $\check{L}$ *on* D if

(i) $\mathbb{P}^x(\Xi_0 = x) = 1$,
(ii) $\mathbb{P}^x(\Xi_t \in D, \forall t \geq 0) = 1$, and
(iii) $(f(\Xi_t) - f(\Xi_0) - \int_0^t \check{L} f(\Xi_u)\, du; (\mathcal{B}_t)_{t\geq 0})$ is a $\mathbb{P}^x$-martingale for all $f \in C_0^2(D)$ and each $x \in D$, where $C_0^2(D)$ is the class of twice continuously differentiable functions with compact support in D.

Let $\xi = (z, y) \in \mathbb{R}^d \times E$ and consider the generator

$$L := \frac{1}{2}\sum_{i,j=1}^{d+1} \widetilde{A}_{ij}(\xi)\frac{\partial^2}{\partial \xi_i \partial \xi_j} + \sum_{i=1}^{d+1} \widetilde{b}_i(\xi)\frac{\partial}{\partial \xi_i}, \quad \widetilde{A} := \begin{pmatrix} \Sigma & \Upsilon \\ \Upsilon' & A \end{pmatrix}, \; \widetilde{b} := \begin{pmatrix} \mu \\ b \end{pmatrix}. \quad (2.10)$$

This is the infinitesimal generator of (R, Y) from (2.7) and (2.8). Assumptions 2.8 and 2.9 imply that there exists a unique solution $(\mathbb{P}^\xi)_{\xi \in \mathbb{R}^d \times E}$ on $(\Omega, \mathcal{F})$ to the martingale problem on $\mathbb{R}^d \times E$ for L; see [46, Theorem 1.12.1].[6]

Remark 2.11 There is a one-to-one correspondence between solutions to the martingale problem and weak solutions for (R, Y); see [47, Chap. V]. In fact, under the given assumptions, there exist 1- and d-dimensional $\mathbb{P}^\xi$-independent Brownian motions W and B such that B is $\mathbb{P}^\xi$-independent of Y and such that the tuple $((R, Y), (W, Z), (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}^\xi))$ is a weak solution of (2.7) and (2.8), where $Z = \rho W + \bar{\rho} B$.

2.3.2 *The value function*

Recall that we assume $p \neq 0$. Let H be an admissible strategy and $\pi = (\pi^1, \ldots, \pi^d)'$ the corresponding risky weights, $\pi^i = H^i S^i / X^H$ for $i = 1, \ldots, d$, and write X^π for X^H so that

$$\frac{dX_t^\pi}{X_t^\pi} = r(Y_t)\, dt + \pi_t'\, dR_t.$$

The value function for the horizon $T \in \mathbb{R}_+$ is given, for $0 \leq t \leq T$, $y \in E$, by

$$u^T(t, x, y) := \sup_{\pi \text{ admissible}} \mathbb{E}^{\mathbb{P}^{y_0}}\left[(X_T^\pi)^p / p \,\middle|\, X_t = x, Y_t = y\right], \quad (2.11)$$

where $y_0 \in E$ is some arbitrary point (which is inconsequential since (R, Y) has the Markov property). It is understood that $u^{0,T} = u^T(0, 1, y)$.[7] To account for the homogeneity of power utility, define the *reduced* value function v^T via

$$u^T(t, x, y) = \frac{x^p}{p}\left(v^T(t, y)\right)^\delta, \quad \text{where } \delta := \frac{1}{1 - q\rho'\rho}, \; q := \frac{p}{p-1}. \quad (2.12)$$

2.3.3 *Heuristics for the classic turnpike*

The next heuristic argument shows why under certain technical conditions, the classic turnpike is expected to hold in the present diffusion setting. Suppose that u^T from

[6] Since $R_0 = 0$ by assumption, $\mathbb{P}^\xi$ with $\xi = (0, y)$ is denoted as $\mathbb{P}^y$. The same convention applies to other probabilities introduced later.

[7] In the model (2.7) and (2.8), $u^{0,T}$ depends on the initial value of the state variable $Y_0 = y$. Hence u^T is a function of y. Since Proposition 2.5 reduces the problem to the comparison of the optimal isoelastic finite-horizon portfolio with its long-run limit, the superscript 0 will be omitted in this section.

(2.11) is smooth. The power transformation in (2.12) linearizes the HJB equation satisfied by u^T (see Zariphopoulou [49]), so that v^T is expected to solve

$$\begin{aligned} -\partial_t v &= \mathcal{L}v + c\,v, \quad (t,y) \in (0,T) \times E, \\ v(T,y) &= 1, \quad y \in E, \end{aligned} \tag{2.13}$$

where

$$\mathcal{L} := \frac{1}{2} A\,\partial^2_{yy} + B\,\partial_y, \quad B := b - q\Upsilon'\Sigma^{-1}\mu, \quad c := \frac{1}{\delta}\left(pr - \frac{q}{2}\mu'\Sigma^{-1}\mu\right). \tag{2.14}$$

Moreover, the optimal portfolio for the horizon T problem is (with all functions evaluated at (t, Y_t)):

$$\pi^T = \frac{1}{1-p}\Sigma^{-1}\left(\mu + \delta\Upsilon\frac{v^T_y}{v^T}\right). \tag{2.15}$$

For $Y_0 = y$, the wealth process corresponding to this portfolio leads to the optimal terminal wealth $X^{\pi^T}_T$, which in turn defines the probability $\mathbb{P}^{T,y}$ by (2.5). A calculation shows that the density of $\mathbb{P}^{T,y}$ with respect to $\mathbb{P}^y$ on $\mathcal{F}_t$ is

$$\begin{aligned} D^{v^T}_t &:= \left.\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}\right|_{\mathcal{F}_t} \\ &= \mathcal{E}\left(\int\left(-q\Upsilon'\Sigma^{-1}\mu + A\frac{v^T_y}{v^T}\right)\frac{1}{a}\,dW \right. \\ &\quad \left. - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{v^T_y}{v^T}\right)'\sigma\bar{\rho}\,dB\right)_t, \end{aligned} \tag{2.16}$$

where the independent Brownian motions (W, B) are as in Remark 2.11. In view of Lemma 2.6, identifying the limiting behavior of the density $D^{v^T}_t$ as $T \to \infty$ is crucial to pass from the abstract to the classic turnpike. A guess for the limiting density is obtained from the *ergodic* version of (2.13):

$$\lambda\,v = \mathcal{L}\,v + c\,v, \quad y \in E. \tag{2.17}$$

The principal (i.e., with $v > 0$) solution $(\hat{v}, \lambda_c)$ controls the long-run limit of the utility maximization problem (see [21]) and is related to v^T by $v^T(t,y) \sim e^{\lambda_c(T-t)}\hat{v}(y)$ as $T \to \infty$. The long-run portfolio is given by

$$\hat{\pi} = \frac{1}{1-p}\Sigma^{-1}\left(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}}\right). \tag{2.18}$$

Define the process

$$D_t^{\hat{v}} := \mathcal{E}\left(\int\left(-q\Upsilon'\Sigma^{-1}\mu + A\frac{\hat{v}_y}{\hat{v}}\right)\frac{1}{a}\,dW \right.$$
$$\left. - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{\hat{v}_y}{\hat{v}}\right)'\sigma\bar{\rho}\,dB\right)_t. \tag{2.19}$$

Assume there exists a long-run probability $\hat{\mathbb{P}}^y$ on $(\Omega, \mathcal{F})$ whose density with respect to $\mathbb{P}^y$ is given by $D_t^{\hat{v}}$ for $t \geq 0$. Then the density of $\mathbb{P}^{T,y}$ with respect to $\hat{\mathbb{P}}^y$ on $\mathcal{F}_t$ takes the form (see (4.12) below):

$$\left.\frac{d\mathbb{P}^{T,y}}{d\hat{\mathbb{P}}^y}\right|_{\mathcal{F}_t} = \frac{D_t^{v^T}}{D_t^{\hat{v}}} = \frac{h^T(t, Y_t)}{h^T(0, y)}\,\mathcal{E}\left(-\int q\delta a\frac{h_y^T}{h^T}\bar{\rho}'\rho\,d\hat{B}\right)_t, \tag{2.20}$$

where $\hat{B}$ is a $\hat{\mathbb{P}}^y$-Brownian motion and h^T is the ratio between v^T and its long-run analog $\hat{v}$, i.e., h^T satisfies $v^T(t, y) = e^{\lambda_c(T-t)}\hat{v}(y)h^T(t, y)$. Given that v^T and $(\hat{v}, \lambda_c)$ satisfy (2.13) and (2.17), respectively, h^T satisfies the PDE

$$\begin{aligned} \partial_t h^T + \mathcal{L}^{\hat{v},0}h^T &= 0, \quad (t, y) \in (0, T) \times E, \\ h^T(T, y) &= \frac{1}{\hat{v}(y)}, \quad y \in E, \end{aligned} \tag{2.21}$$

where

$$\mathcal{L}^{\hat{v},0} := \mathcal{L} + A\frac{\hat{v}_y}{\hat{v}}\partial_y \tag{2.22}$$

is the differential part of the *h-transform* of $\mathcal{L}$ using $\hat{v}$ as the h-function; see [46, Sect. 4.1]. Since $\mathcal{L}^{\hat{v},0}$ is the infinitesimal generator of Y under $\hat{\mathbb{P}}^y$, one expects from the Feynman–Kac formula that h^T has the stochastic representation

$$h^T(t, y) = \mathbb{E}^{\hat{\mathbb{P}}^y}\left[\frac{1}{\hat{v}(Y_{T-t})}\right], \quad (t, y) \in [0, T] \times E. \tag{2.23}$$

If Y is positive recurrent under $(\hat{\mathbb{P}}^y)_{y\in E}$ and if $1/\hat{v}$ is integrable with respect to the invariant density, (2.23) in conjunction with the ergodic theorem implies that $h^T(t, y)$ converges to a constant for all (t, y) as $T \to \infty$. Hence, (2.20) implies that $d\mathbb{P}^{T,y}/d\hat{\mathbb{P}}^y|_{\mathcal{F}_t}$ converges to 1 in $\hat{\mathbb{P}}^y$-probability, and the classic turnpike theorem follows from Lemma 2.6.

Remark 2.12 For multivariate factor models with stochastic correlations, consider the reduced value v^T defined by $u^T(t, x, y) = (x^p/p)\exp(v^T(t, y))$. Then v^T is expected to solve a semilinear HJB equation with quadratic nonlinearity in the first-order derivative. The associated ergodic HJB equation has been studied in [21, 30]. Given its solution $(\hat{v}, \lambda_c)$, the function $h^T(t, y) = v^T(t, y) - \lambda_c(T - \lambda) - \hat{v}(y)$ is expected to solve another semilinear equation with modified first-order derivative term, which is similar to the h-transform above. Since the study of semilinear PDEs requires a different set of techniques, turnpike theorems for multivariate diffusion models are deferred to a separate treatment.

2.3.4 The classic turnpike theorem

Turning the previous argument into a precise statement requires some hypotheses. First, the following assumption ensures the existence of a principal solution $(\hat{v}, \lambda_c)$ to the ergodic HJB equation in (2.17) such that Y is positive recurrent under $(\hat{\mathbb{P}}^y)_{y\in E}$ and such that $1/\hat{v}$ is integrable with respect to the invariant density for Y.

Assumption 2.13 There exist $(\hat{v}, \lambda_c)$ such that $\hat{v} \in C^2(E)$, $\hat{v} > 0$, and $(\hat{v}, \lambda_c)$ solves Eq. (2.17). For the $y_0 \in E$ in Assumption 2.9,[8]

$$\int_\alpha^{y_0} \frac{1}{\hat{v}^2 A\hat{m}(y)}\, dy = \infty, \qquad \int_{y_0}^{\beta} \frac{1}{\hat{v}^2 A\hat{m}(y)}\, dy = \infty, \tag{2.24}$$

$$\int_\alpha^\beta \hat{v}^2\, \hat{m}(y)\, dy = 1, \qquad \int_\alpha^\beta \hat{v}\, \hat{m}(y)\, dy < \infty, \tag{2.25}$$

where

$$\hat{m}(y) := \frac{1}{A(y)} \exp\left(\int_{y_0}^{y} \frac{2B(z)}{A(z)}\, dz\right). \tag{2.26}$$

Remark 2.14 By the linearity of (2.17), assuming $\int_\alpha^\beta \hat{v}^2 \hat{m}(y)\, dy = 1$ is equivalent to assuming $\int_\alpha^\beta \hat{v}^2 \hat{m}(y)\, dy < \infty$ since $\hat{v}$ may be renormalized.

A simple criterion to check Assumption 2.13 is the following.

Proposition 2.15 *Let Assumptions 2.8 and 2.9 hold. Suppose that c and $\hat{m}$ satisfy*

$$\int_\alpha^\beta \hat{m}(y)\, dy < \infty, \tag{2.27}$$

$$\lim_{y\downarrow\alpha} c(y) = \lim_{y\uparrow\beta} c(y) = -\infty. \tag{2.28}$$

Then Assumption 2.13 holds.

Remark 2.16 If the interest rate r is bounded from below and $p < 0$, (2.28) states that the squared norm of the vector of risk premia $\sigma^{-1}\mu$ goes to ∞ at the boundary of the state space E. Even though Proposition 2.15 may not be applicable, Assumption 2.13 holds also for $0 < p < 1$, under some parameter restrictions; see Examples 2.22 and 2.23 below.

[8] Any $y_0 \in E$ suffices. This y_0 is chosen to align m with $\hat{m}$.

To understand (2.24) and (2.25), define the operator

$$\hat{\mathcal{L}} := \frac{1}{2}\sum_{i,j=1}^{d+1} \widetilde{A}_{ij}(\xi)\frac{\partial^2}{\partial\xi_i\partial\xi_j} + \sum_{i=1}^{d+1}\hat{b}_i(\xi)\frac{\partial}{\partial\xi_i}, \qquad \hat{b} := \begin{pmatrix} \frac{1}{1-p}(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}}) \\ B + A\frac{\hat{v}_y}{\hat{v}} \end{pmatrix}, \quad (2.29)$$

where $\widetilde{A}$ is from (2.10). Condition (2.24) implies that the martingale problem for $\hat{\mathcal{L}}$ on $\mathbb{R}^d \times E$ has a unique solution $(\hat{\mathbb{P}}^\xi)_{\xi\in\mathbb{R}^d\times E}$ and that $\hat{\mathbb{P}}^\xi$ is equivalent to $\mathbb{P}^\xi$ (see Lemma 4.2 below). The family $(\hat{\mathbb{P}}^\xi)_{\xi\in\mathbb{R}^d\times E}$ is called the *long-run probability*. The measure $\hat{\mathbb{P}}^\xi$ with $\xi = (0, y)$ is the measure $\hat{\mathbb{P}}^y$ in the heuristic argument. Proposition 4.6 below shows that $d\hat{\mathbb{P}}^y/d\mathbb{P}^y|_{\mathcal{F}_t} = D_t^{\hat{v}}$, and for all $T > 0$ it both constructs a strictly positive classical solution v^T to (2.13) and verifies that the value function u^T in (2.11) can be represented as $u^T(t, x, y) = (x^p/p)(v^T(t, y))^\delta$.

Conditions (2.24) and the identity in (2.25) imply that Y is ergodic under $(\hat{\mathbb{P}}^y)_{y\in E}$ with the invariant density $\hat{v}^2\hat{m}$ (see Lemma 4.1 below and Sect. 4.2 for a precise definition of ergodicity). The inequality in (2.25) ensures that $1/\hat{v}$ is integrable with respect to the invariant density. Hence h^T, with its stochastic representation (2.23), converges to a constant as $T \to \infty$. Combined with the representation (2.20), we get the following:

Lemma 2.17 *Let Assumptions 2.8, 2.9 and 2.13 hold. Then, for all $y \in E$ and $t, \varepsilon > 0$,*

$$\lim_{T\to\infty} \hat{\mathbb{P}}^y\left(\left|\frac{d\mathbb{P}^{T,y}}{d\hat{\mathbb{P}}^y}\bigg|_{\mathcal{F}_t} - 1\right| \geq \epsilon\right) = 0. \quad (2.30)$$

In light of Lemma 2.6, the classic turnpike follows under additional assumptions on r, Y, E which enforce Assumption 2.2; for example, one can take $\underline{r} \leq r(y) \leq \overline{r}$ for some constants $0 < \underline{r} \leq \overline{r}$.

Theorem 2.18 (Classic turnpike) *Let Assumptions 2.1–2.4, 2.8, 2.9 and 2.13 hold. Then, for all $y \in E$, $0 \neq p < 1$ and $\epsilon, t > 0$:*

(a) $\lim_{T\to\infty} \mathbb{P}^y(\sup_{u\in[0,t]} |r_u^T - 1| \geq \epsilon) = 0$;

(b) $\lim_{T\to\infty} \mathbb{P}^y([\Pi^T, \Pi^T]_t \geq \epsilon) = 0$.

2.3.5 The explicit turnpike theorem

Abstract and classic turnpikes compare the finite-horizon optimal portfolio of a generic utility to that of its CRRA benchmark at the same finite horizon. However, the finite-horizon optimal portfolio π^T for the CRRA benchmark still depends on the horizon T. By contrast, the explicit turnpike, discussed next, uses $\hat{\pi}$ in (2.18) as the benchmark. Unlike π^T, the portfolio $\hat{\pi}$ is myopic.

This result has two main implications. First, and most importantly, it shows that the two approximations of replacing a generic utility with a power utility, and a finite-horizon problem with its long-run limit, lead to small errors as the horizon becomes large. Second, this theorem has a non-trivial statement even for U in the CRRA class;

in this case, it states that the optimal finite-horizon portfolio converges to the long-run optimal portfolio, identified as a solution to the ergodic HJB equation (2.17).

To state the explicit turnpike, define, in analogy to (2.4), the ratio of optimal wealth processes relative to the long-run benchmark, and their stochastic logarithms, as

$$\hat{r}_u^T := \frac{X_u^{1,T}}{\hat{X}_u}, \qquad \hat{\Pi}_u^T := \int_0^u \frac{d\hat{r}_v^T}{\hat{r}_{v-}^T}, \quad \text{for } u \in [0, T],$$

where $\hat{X}$ is the wealth process of the long-run portfolio $\hat{\pi}$.

Theorem 2.19 (Explicit turnpike) *Under the assumptions of Theorem 2.18, for any $y \in E$, $\epsilon, t > 0$ and $0 \neq p < 1$:*

(a) $\lim_{T\to\infty} \mathbb{P}^y (\sup_{u\in[0,t]} |\hat{r}_u^T - 1| \geq \epsilon) = 0$;
(b) $\lim_{T\to\infty} \mathbb{P}^y ([\hat{\Pi}^T, \hat{\Pi}^T]_t \geq \epsilon) = 0$.

If U is in the CRRA class, then (2.3) is not needed for the above convergence.

When U is in the CRRA class, consider the portfolio π^T, optimal for the horizon T, and its long-run limit $\hat{\pi}$ in the feedback forms $\pi_t^T = \pi^T(t, Y_t)$ and $\hat{\pi}_t = \hat{\pi}(Y_t)$. The following convergence result for the optimal strategies can be obtained as a direct consequence of the fact that $h^T(t, y)$ converges towards a constant and that furthermore $h_y^T(t, y)/h^T(t, y)$ converges to 0 as $T \to \infty$.

Corollary 2.20 *Let the assumptions of Theorem 2.18 hold without (2.3), for any $t \geq 0$ and $p < 1$. Then $\lim_{T\to\infty} \pi^T(t, y) = \hat{\pi}(y)$, locally uniformly in $[0, \infty) \times E$.*

2.4 Applications

Before proving the main results, three examples of their significance are offered. First is an application to target-date mutual funds and the social planner problem.

Example 2.21 Consider several investors who differ in their initial capitals $(x_i)_{i=1}^n$ and risk aversions $(\gamma_i)_{i=1}^n$ but share the same long horizon T. Suppose that they do not invest independently but rather pool their wealth into a common fund, delegate a manager to invest it, and then collect the proceeds on their respective capitals under the common investment strategy. This setting is typical of target-date retirement funds, in which savings from a diverse pool of participants are managed according to a single strategy, characterized by the common horizon T.

Suppose the manager invests to maximize a weighted sum of the investors' expected utilities, thereby solving the problem

$$\max_{X\in\mathcal{X}^T} \sum_{i=1}^n w_i \mathbb{E}^{\mathbb{P}}\left[\frac{(x_i X_T)^{1-\gamma_i}}{1-\gamma_i}\right]$$

for some positive $(w_i)_{i=1}^n$. By homogeneity and linearity, this problem is equivalent to maximizing the expected value $\mathbb{E}^{\mathbb{P}}[U(X_T)]$ of the master utility function[9]

$$U(x) = \sum_{i=1}^{n} \tilde{w}_i \frac{x^{1-\gamma_i}}{1-\gamma_i}, \quad \text{where } \tilde{w}_i = w_i x_i^{1-\gamma_i}.$$

Thus, the fund manager is akin to a social planner, who ponders the welfare of various investors according to the weights $\tilde{w}_i$. The question is how these weights affect the choice of the common fund's strategy if the horizon is distant, as for most retirement funds.

While this problem is intractable for a fixed horizon T, turnpike theorems offer a crisp solution in the long-run limit. Indeed, the master utility function satisfies Assumption 2.1 with $\gamma = 1 - p = \min_{i=1,\dots,n} \gamma_i$. Thus, for any market that satisfies additionally Assumptions 2.2–2.4, it is optimal for the fund manager to act on behalf of the least risk-averse investor.

The implication is that most or nearly all fund participants will find that the fund takes more risk than they would like, *regardless* of the welfare weights $\tilde{w}_i$ (provided that they are strictly positive). The result holds irrespective of market completeness or independence of returns, and indicates that a social planner objective is ineffective in choosing a portfolio that balances the needs of investors with different preferences.

Note that this result points in the same direction as the ones of Benninga and Mayshar [1] and Cvitanić and Malamud [8], with the crucial difference that prices are endogenous in their models, while they are exogenous in our setting. Finally, the result should be seen in conjunction with the classic numéraire property of the log-optimal portfolio, whereby the wealth process of the logarithmic investor becomes arbitrarily larger than any other wealth process. In spite of this property, the fund manager does not choose the log-optimal strategy, but the one optimal for the least risk-averse investor.

In the next example, returns of risky assets have constant volatility, but their drift is a correlated Ornstein–Uhlenbeck process. The optimal CRRA portfolios are neither myopic (except when $p = 0$) nor have independent returns (even when $p = 0$) and hence Corollary 2.7 is not applicable. Yet, both the classic and the explicit turnpikes hold in this model, in the form of Theorems 2.18 and 2.19, even for $0 < p < 1$.

Example 2.22 Consider the diffusion model

$$dR_t = Y_t\, dt + dZ_t \quad \text{and} \quad dY_t = -Y_t\, dt + dW_t.$$

The correlation ρ takes values in $(-1, 1)$ and the safe rate is a constant $r > 0$. Clearly, Assumptions 2.8 and 2.9 hold. Furthermore, for p, ρ satisfying $2p(1 + \rho) < 1$ (or, equivalently, $1 + q(2\rho + 1) > 0$ where $q = p/(p - 1)$), Assumption 2.13 holds as

[9] If a logarithmic investor is present ($\gamma_i = 1$ for some i), a constant is added to $U(x)$, and the stated equivalence remains valid.

well. Indeed, it can be directly verified that

$$\hat{v}(y) = \hat{v}(0)e^{\frac{1}{2}(1+q\rho-\sqrt{1+q(2\rho+1)})y^2}, \qquad \lambda_c = \frac{pr}{\delta} + \frac{1}{2}\big(1 + q\rho - \sqrt{1 + q(2\rho + 1)}\big).$$

Therefore, Theorems 2.18 and 2.19 follow. Note that $2p(1 + \rho) < 1$ always holds when either $p < 0$ or $\rho < -1/2$, but it may also hold when $0 < p < 1$. Thus even though the hypotheses of Proposition 2.15 are not met, the turnpike theorems still follow.

It is well known that for $0 < p < 1$, it can happen that the value function $v^T(t, y)$ explodes in finite time. Indeed, in the present setup, v^T takes the form $v^T(t, y) = e^{C(T-t)-(y^2/2)A(T-t)}$, where $A(s)$ solves the Riccati ODE

$$\dot{A}(s) = -A^2(s) - 2(1 + q\rho)A(s) + \frac{q}{\delta}, \qquad A(0) = 0,$$

and where, given A, $C(s) = (pr/\delta)s - \frac{1}{2}\int_0^s A(u)\,du$. When $2p(1 + \rho) \le 1$, one can show that the solution $A(s)$ remains finite for all s. However, for $2p(1+\rho) > 1$, $A(s)$ explodes to $-\infty$ as s approaches s^*, the minimal positive solution of

$$\tan\Big(s\sqrt{-\big(1 + q(2\rho + 1)\big)}\Big) = -\frac{\sqrt{-(1 + q(2\rho + 1))}}{1 + q\rho},$$

with the convention that $\tan y = \infty$ implies that $y = \pi/2$. Therefore, except for the boundary case $2p(1 + \rho) = 1$, turnpike theorems hold when the value function remains finite for all T, and vice versa.

Consider the case $\rho = 0$. Since $\Upsilon = 0$, the optimal portfolio for a CRRA investor is the myopic portfolio $\pi_t^T = Y_t/(1 - p)$; see (2.15). However, the density $d\mathbb{P}^{T,y}/d\mathbb{P}^y|_{\mathcal{F}_t}$ depends on the horizon T. Indeed, it follows by plugging into (2.16) that

$$\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}\bigg|_{\mathcal{F}_t} = \mathcal{E}\bigg(\int \frac{v_y^T(s, Y_s)}{v^T(s, Y_s)}\,dW_s - q\int Y_s\,dZ_s\bigg)_t,$$

where v^T satisfies the HJB equation

$$\partial_t v + \frac{1}{2}\partial_{yy}^2 v - y\partial_y v + \Big(rp - \frac{q}{2}y^2\Big)v = 0$$

with $v(T, y) = 1$. The above density is independent of T only if the function $g^T(t, y) := v_y^T(t, y)/v^T(t, y)$ is independent of T for any fixed (t, y). It can be shown that v^T is smooth, and not just twice continuously differentiable, in the state variable y, and hence g^T satisfies

$$\partial_t g + \frac{1}{2}\partial_{yy}^2 g + (g - y)\partial_y g - g - qy = 0$$

with $g(T, y) = 0$. If g^T were independent of T, 0 would be a solution to the previous equation. However, this is clearly not the case for $q \neq 0$.

In the final example (cf. Guasoni and Robertson [21, 22]), a single state variable follows the square-root diffusion of Feller [14], and simultaneously affects the interest rate, the volatilities of risky assets, and the Sharpe ratios. This model is not necessarily affine, yet both the classic and explicit turnpikes always hold for $p < 0$. Furthermore, when the model is restricted to be affine ($\mu_0 = 0$ below), turnpikes may hold even for $0 < p < 1$.

Example 2.23 Consider the diffusion model

$$dR_t = (\mu_0 + \mu_1 Y_t)\,dt + \sqrt{Y_t}\,dZ_t \quad \text{and} \quad dY_t = b(\theta - Y_t)\,dt + a\sqrt{Y_t}\,dW_t. \tag{2.31}$$

The correlation ρ takes values in $(-1, 1)$ and the safe rate is a constant $r > 0$. Note that for $\mu_0 \neq 0$, this model is not affine. Clearly, Assumption 2.8 holds. In order to make the original model well posed (i.e., Assumption 2.9 holds), it is assumed that $b, \theta, a > 0$ and $b\theta - a^2/2 \geq 0$. Set

$$\Lambda := \big(b\theta - a^2/2 - qa\rho\mu_0\big)^2 + a^2 q\mu_0^2/\delta, \qquad \Theta := (b + qa\rho\mu_1)^2 + a^2 q\mu_1^2/\delta.$$

If $\Lambda \geq 0$, $\Theta > 0$ (which is always the case when $p < 0$, $\mu_1 \neq 0$), the candidate $(\hat{v}, \lambda_c)$ is given by (see [21]):

$$\hat{v}(y) = \hat{v}(1) y^{(1/a^2)(e\sqrt{\Lambda} - (b\theta - a^2/2 - qa\rho\mu_0))} e^{(1/a^2)(b + qa\rho\mu_1 - \sqrt{\Theta}(y-1))},$$

$$\lambda_c = \frac{1}{\delta}\bigg(pr - q\mu_0\mu_1 + \big(\delta/a^2\big)\Big((b + qa\rho\mu_1)(b\theta - qa\rho\mu_0) - \sqrt{\Theta}\big(\sqrt{\Lambda} + a^2/2\big)\Big)\bigg).$$

Indeed, in Assumption 2.13, $(\hat{v}, \lambda_c)$ satisfy (2.17), (2.24) and the first equality in (2.25), where now $\hat{m}(y) = \hat{m}(1) y^{(2/a^2)(b\theta - a^2/2 - qa\rho\mu_0)} e^{-(2/a^2)(b + qa\rho\mu_1)(y-1)}$. As for the inequality in (2.25), note that for some positive $\tilde{C}$, we have

$$\hat{v}(y)\hat{m}(y) = \tilde{C} y^{\mathbf{A}} e^{-\mathbf{B}y},$$

$$\mathbf{A} = \big(\sqrt{\Lambda} + (b\theta - a^2/2 - qa\rho\mu_0)\big)/a^2,$$

$$\mathbf{B} = \big(\sqrt{\Theta} + (b + qa\rho\mu_1)\big)/a^2.$$

The inequality in (2.25) thus holds if $\mathbf{A} > -1$ and $\mathbf{B} > 0$. When $p < 0$, $\mu_1 \neq 0$, this is always the case. For $0 < p < 1$, this is a very delicate parameter restriction which simplifies if $\mu_1 \neq 0$, $\mu_0 = 0$ or, equivalently, if the model is affine. Summarizing:

Lemma 2.24 *In the model* (2.31), *assume that* $\rho \in (-1, 1)$, $r, b, \theta, a > 0$, $b\theta \geq a^2/2$. *Then Assumptions* 2.8 *and* 2.9 *hold. Furthermore*:

(i) *If* $p < 0$ *and* $\mu_1 \neq 0$, *then* $\Lambda \geq 0$, $\Theta > 0$, $\mathbf{A} > -1$ *and* $\mathbf{B} > 0$. *Thus, Assumption* 2.13 *holds and the classic and explicit turnpike theorems follow.*

(ii) *If* $0 < p < 1$, *then Assumption* 2.13 *holds, and hence the turnpike theorems follow, provided that* $\Lambda \geq 0$, $\Theta > 0$, $\mathbf{A} > -1$ *and* $\mathbf{B} > 0$. *When* $\mu_0 = 0$, $\mu_1 \neq 0$, *the condition* $\Lambda \geq 0$, $\Theta > 0$, $\mathbf{A} > -1$, $\mathbf{B} > 0$ *is equivalent to*

$$(b + qa\rho\mu_1)^2 + a^2 q\mu_1^2/\delta > 0, \qquad b + qa\rho\mu_1 > 0.$$

3 Proof of the abstract turnpike

This section contains the results leading to the abstract version of the turnpike theorem. The proof proceeds through two main steps:

(i) Establishing that optimal payoffs for the generic utility converge to their CRRA counterparts.
(ii) Obtaining from the convergence of optimal payoffs the convergence of wealth processes.

These steps are taken in Dybvig et al. [12] to prove turnpike theorems in complete markets. Two new techniques employed here allow us to analyze incomplete markets: (i) the measure is changed to $\mathbb{P}^T$ (the numéraire is changed to $X^{0,T}$), so that r^T is a $\mathbb{P}^T$-supermartingale on $[0, T]$; (ii) a novel estimate using the first-order condition (see Lemma 3.6) is introduced. The convergence of $\mathbb{E}^{\mathbb{P}^T}[|r^T - 1|]$ is derived using this estimate. However, additional technical difficulties come along with these new techniques. In particular, every quantity, especially the measure $\mathbb{P}^T$, depends on the horizon T.

3.1 Convergence of optimal payoffs

First, note that Assumption 2.3 implies the existence of a *deflator*, that is, a strictly positive process Y such that YX is a (nonnegative) supermartingale on $[0, T]$ for all $X \in \mathcal{X}$ and $T > 0$. Condition (2.3) entails that $\lim_{T\to\infty} \mathbb{E}[Y_T] = 0$ for any such deflator Y. In this section, the capital letter Y is used for deflators, while in the section on diffusion models it denotes the state variable. Recall a result from [33]:

Theorem 3.1 (Karatzas–Žitković) *Under Assumptions* 2.1–2.4, *the optimal payoffs are*

$$X_T^{i,T} = I^i\big(y^{i,T} Y_T^{i,T}\big), \quad i = 0, 1,\ T > 0, \tag{3.1}$$

where I^0 *is the inverse function of* x^{p-1}, I^1 *is the inverse function of* $U'(x)$, *the positive constant* $y^{i,T}$ *is a Lagrangian multiplier, and* $Y^{i,T}$ *is some supermartingale deflator. Moreover,*

$$y^{i,T} = \mathbb{E}^{\mathbb{P}}\big[(U^i)'\big(X_T^{i,T}\big)X_T^{i,T}\big] \geq \mathbb{E}^{\mathbb{P}}\big[(U^i)'\big(X_T^{i,T}\big)X_T\big], \quad i = 0, 1,\ T > 0, \tag{3.2}$$

for any $X \in \mathcal{X}$. *Here* $U^0(x) = x^p/p$ *and* $U^1(x) = U(x)$.

Remark 3.2 (i) It follows from (3.1) and the Inada condition that $X_T^{i,T} > 0$ $\mathbb{P}$-a.s. for $i = 0, 1$ and $T \geq 0$. Since $X^{i,T}$ is a nonnegative $\mathbb{Q}^T$-supermartingale and $\mathbb{Q}^T$ is equivalent to $\mathbb{P}$, it follows that $X_t^{i,T} > 0$ $\mathbb{P}$-a.s. for $0 \leq t \leq T$.

(ii) Condition (2.3) entails that $\lim_{T\to\infty} \mathbb{P}^T(S_T^0 \geq N) = 1$ for any $N > 0$ and $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}}[Y_T^{i,T}] = 0$ for $i = 0, 1$.

(iii) Recall the probability measure $\mathbb{P}^T$ defined in (2.5). The optimal wealth process $X^{0,T}$ has the numéraire property under $\mathbb{P}^T$, i.e., $\mathbb{E}^{\mathbb{P}^T}[X_T/X_T^{0,T}] \leq 1$ for any $X \in \mathcal{X}$. This claim follows from $\mathbb{E}^{\mathbb{P}}[(X_T^{0,T})^p(X_T/X_T^{0,T} - 1)] \leq 0$, obtained from (3.2), and switching the expectation from $\mathbb{P}$ to $\mathbb{P}^T$.

Both $X_T^{0,T}$ and $X_T^{1,T}$ will be shown to be unbounded as $T \to \infty$. However, the main result of this subsection, Proposition 3.8, shows that their ratio at the horizon T, given by r_T^T from (2.4), satisfies $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}[|r_T^T - 1|] = 0$. Proposition 3.8 will be the culmination of a series of auxiliary results.

Note: Assumptions 2.1–2.4 are enforced in the rest of this subsection.

Lemma 3.3

$$\lim_{T\to\infty} \mathbb{P}^T\big(X_T^{0,T} \geq N\big) = 1, \quad \textit{for any } N > 0.$$

Proof It suffices to prove $\limsup_{T\to\infty} \mathbb{P}^T(X_T^{0,T} < N) = 0$ for each fixed N. To this end, the numéraire property of $X^{0,T}$ under $\mathbb{P}^T$ implies that

$$1 \geq \mathbb{E}^{\mathbb{P}^T}\left[\frac{S_T^0}{X_T^{0,T}}\right] \geq \mathbb{E}^{\mathbb{P}^T}\left[\frac{S_T^0}{X_T^{0,T}} 1_{\{X_T^{0,T} < N, S_T^0 \geq \tilde{N}\}}\right] \geq \frac{\tilde{N}}{N} \mathbb{P}^T\big(X_T^{0,T} < N, S_T^0 \geq \tilde{N}\big),$$

for any positive constant $\tilde{N}$. As a result, $\mathbb{P}^T(X_T^{0,T} < N, S_T^0 \geq \tilde{N}) \leq N/\tilde{N}$. Combining the last inequality with Remark 3.2(ii), it follows that

$$\begin{aligned}\limsup_{T\to\infty} \mathbb{P}^T\big(X_T^{0,T} < N\big) &\leq \limsup_{T\to\infty} \mathbb{P}^T\big(X_T^{0,T} < N, S_T^0 \geq \tilde{N}\big) + \lim_{T\to\infty} \mathbb{P}^T\big(S_T^0 < \tilde{N}\big) \\ &\leq N/\tilde{N}.\end{aligned}$$

Then, the statement follows since $\tilde{N}$ is chosen arbitrarily. □

Recall the Lagrangian multipliers $y^{i,T}$, $i = 0, 1$, from Theorem 3.1. The following result presents the asymptotic behavior of $y^{0,T}/y^{1,T}$ as $T \to \infty$.

Lemma 3.4

$$\liminf_{T\to\infty} \frac{y^{0,T}}{y^{1,T}} \geq 1.$$

Proof We argue separately for $p = 0$, $p \in (0, 1)$, and $p < 0$. Throughout this proof, in order to ease notation, set $\alpha_T = y^{1,T}$, $Y_T = Y_T^{1,T}$, $\widetilde{Y}_T = Y_T^{0,T}$, $X_T = X_T^{1,T}$,

$\widetilde{X}_T = X_T^{0,T}$, and $I = (U')^{-1}$. All expectations are under $\mathbb{P}$. Observe first that

$$\lim_{y \downarrow 0} I(y) y^{\frac{1}{1-p}} = 1. \tag{3.3}$$

Indeed, set $x = I(y)$, so that $x \uparrow \infty$ as $y \downarrow 0$. Then the convergence above follows from (2.2) via

$$\frac{I(y)}{y^{\frac{1}{p-1}}} = \frac{I(U'(x))}{(U'(x))^{\frac{1}{p-1}}} = \frac{x}{(U'(x))^{\frac{1}{p-1}}} = \left(\frac{x^{p-1}}{U'(x)}\right)^{\frac{1}{p-1}} \longrightarrow 1 \quad \text{as } y \downarrow 0.$$

Case $p = 0$: It follows from (3.3) that for any $\epsilon > 0$, there exists $\delta > 0$ such that $1 - \epsilon \le y I(y) \le 1 + \epsilon$ for $y < \delta$. Then (3.1) and (3.2) imply

$$\begin{aligned}
1 &= \mathbb{E}\big[Y_T I(\alpha_T Y_T)\big] = \mathbb{E}\big[Y_T I(\alpha_T Y_T) 1_{\{\alpha_T Y_T < \delta\}} + Y_T I(\alpha_T Y_T) 1_{\{\alpha_T Y_T \ge \delta\}}\big] \\
&\le \frac{1+\epsilon}{\alpha_T} \mathbb{P}(\alpha_T Y_T < \delta) + I(\delta) \mathbb{E}\big[Y_T 1_{\{\alpha_T Y_T \ge \delta\}}\big] \\
&\le \frac{1+\epsilon}{\alpha_T} + I(\delta) \mathbb{E}[Y_T],
\end{aligned}$$

where the first inequality follows because I is decreasing. Now, the previous inequality combined with Remark 3.2(ii) implies that

$$1 \le \liminf_{T \to \infty} \frac{1+\epsilon}{\alpha_T},$$

from which the statement follows since for $p = 0$, $y^{0,T} = 1$ and ϵ is chosen arbitrarily.

Case $p \in (0, 1)$: It follows from (2.2) that for any $\epsilon > 0$, there exists $M > 0$ such that $1 - \epsilon \le U'(x) x^{1-p} \le 1 + \epsilon$ for $x \ge M$. Then (3.2) implies that

$$\begin{aligned}
1 &= \frac{1}{\alpha_T} \mathbb{E}\big[U'(X_T) X_T\big] \\
&= \frac{1}{\alpha_T} \mathbb{E}\big[U'(X_T) X_T^{1-p} X_T^p 1_{\{X_T \ge M\}}\big] + \frac{1}{\alpha_T} \mathbb{E}\big[U'(X_T) X_T 1_{\{X_T \le M\}}\big] \\
&\le \frac{1+\epsilon}{\alpha_T} \mathbb{E}\big[X_T^p 1_{\{X_T \ge M\}}\big] + \frac{1}{\alpha_T} \mathbb{E}\big[U'(X_T) X_T 1_{\{X_T \le M\}}\big].
\end{aligned}$$

Note that $\frac{1}{\alpha_T} \mathbb{E}[U'(X_T) X_T 1_{\{X_T \le M\}}] = \mathbb{E}[Y_T X_T 1_{\{X_T \le M\}}] \le M \mathbb{E}[Y_T] \to 0$ as $T \to \infty$. Therefore,

$$\frac{1}{1+\epsilon} \le \liminf_{T \to \infty} \frac{1}{\alpha_T} \mathbb{E}\big[X_T^p 1_{\{X_T \ge M\}}\big] \le \liminf_{T \to \infty} \frac{1}{\alpha_T} \mathbb{E}\big[X_T^p\big] \le \liminf_{T \to \infty} \frac{1}{\alpha_T} \mathbb{E}\big[\widetilde{X}_T^p\big],$$

where the third inequality follows from the optimality for $\sup_{x \in \mathcal{X}} \mathbb{E}[X_T^p / p]$ of the process $\widetilde{X} = X^{0,T}$. Note that $y^{0,T} = \mathbb{E}[\widetilde{X}_T^p]$. The statement follows from the previous inequality since ϵ is chosen arbitrarily.

Case $p<0$: For any $\epsilon>0$, there exists $\delta>0$ such that $1-\epsilon\le I(y)y^{\frac{1}{1-p}}\le 1+\epsilon$ for $y<\delta$. Then (3.1) and (3.2) yield (recall that $q=p/(p-1)$ is the exponent conjugate to p):

$$\begin{aligned}1&=\mathbb{E}\big[Y_T I(\alpha_T Y_T)\big]\\&=\mathbb{E}\big[Y_T I(\alpha_T Y_T)\,1_{\{\alpha_T Y_T<\delta\}}\big]+\mathbb{E}\big[Y_T I(\alpha_T Y_T)\,1_{\{\alpha_T Y_T\ge\delta\}}\big]\\&\le\frac{1+\epsilon}{\alpha_T^{\frac{1}{1-p}}}\mathbb{E}\big[Y_T^q\,1_{\{\alpha_T Y_T<\delta\}}\big]+\mathbb{E}\big[Y_T I(\alpha_T Y_T)\,1_{\{\alpha_T Y_T\ge\delta\}}\big].\end{aligned}$$

Since $\mathbb{E}[Y_T I(\alpha_T Y_T)\,1_{\{\alpha_T Y_T\ge\delta\}}]\le I(\delta)\mathbb{E}[Y_T]\to 0$ as $T\to\infty$, the inequality in the last line yields

$$\frac{1}{1+\epsilon}\le\liminf_{T\to\infty}\frac{1}{\alpha_T^{\frac{1}{1-p}}}\mathbb{E}\big[Y_T^q\,1_{\{\alpha_T Y_T<\delta\}}\big]\le\liminf_{T\to\infty}\frac{1}{\alpha_T^{\frac{1}{1-p}}}\mathbb{E}\big[Y_T^q\big].$$

Raising both sides of the previous inequality to the power $1-p$ gives

$$\left(\frac{1}{1+\epsilon}\right)^{1-p}\le\liminf_{T\to\infty}\frac{1}{\alpha_T}\mathbb{E}\big[Y_T^q\big]^{1-p}\le\liminf_{T\to\infty}\frac{1}{\alpha_T}\mathbb{E}\big[\widetilde{X}_T^p\big],$$

from which the statement follows. Since $p<0$, the second inequality above follows from

$$\frac{1}{p}\,\mathbb{E}\big[\widetilde{X}_T^p\big]=\frac{1}{p}\,\mathbb{E}\big[\widetilde{Y}_T^q\big]^{1-p}\le\frac{1}{p}\,\mathbb{E}\big[Y_T^q\big]^{1-p},$$

where the equality holds due to the duality for power utility and the inequality follows from the optimality of $\widetilde{Y}$ for the dual problem which minimizes $\mathbb{E}[-Y_T^q/q]$ among all supermartingale deflators Y. □

The previous two lemmas combined describe the asymptotic behavior of $X_T^{1,T}$ and $\mathfrak{R}(X_T^{1,T})$, where $\mathfrak{R}$ is given in (2.1).

Lemma 3.5

$$\lim_{T\to\infty}\mathbb{P}^T\big(X_T^{1,T}\ge N\big)=1,\quad \textit{for any } N>0.$$

Hence

$$\lim_{T\to\infty}\mathbb{P}^T\big(\big|\mathfrak{R}\big(X_T^{1,T}\big)-1\big|\ge\epsilon\big)=0,\quad \textit{for any } \epsilon>0.$$

Proof It follows from Lemma 3.4 and (3.2) that

$$2\ge\frac{y^{1,T}}{y^{0,T}}\ge\frac{\mathbb{E}^{\mathbb{P}}[X_T^{0,T}U'(X_T^{1,T})]}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,T})^p]}=\mathbb{E}^{\mathbb{P}^T}\left[\frac{U'(X_T^{1,T})}{(X_T^{0,T})^{p-1}}\right]\quad\text{for large } T.\tag{3.4}$$

Combining the previous inequality with Lemma 3.3, the first statement follows. Indeed, for any given M and N, on the set $\{X_T^{1,T} \le N, X_T^{0,T} \ge M\}$, we have both $(X_T^{0,T})^{1-p} \ge M^{1-p}$ and $U'(X_T^{1,T}) \ge U'(N)$; therefore,

$$\begin{aligned} 2 &\ge \mathbb{E}^{\mathbb{P}^T}\left[\frac{U'(X_T^{1,T})}{(X_T^{0,T})^{p-1}} 1_{\{X_T^{1,T} \le N, X_T^{0,T} \ge M\}}\right] \\ &\ge U'(N) M^{1-p}\, \mathbb{P}^T\big(X_T^{1,T} \le N, X_T^{0,T} \ge M\big). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbb{P}^T\big(X_T^{1,T} \le N\big) &\le \mathbb{P}^T\big(X_T^{1,T} \le N, X_T^{0,T} \ge M\big) + \mathbb{P}^T\big(X_T^{0,T} \le M\big) \\ &\le \frac{2}{U'(N)M^{1-p}} + \mathbb{P}^T\big(X_T^{0,T} \le M\big). \end{aligned}$$

Letting first $T \to \infty$ and then $M \to \infty$, the first statement follows.

Now note that for any $\varepsilon > 0$, due to (2.2), there exists N_ε such that $|\mathfrak{R}(x) - 1| < \varepsilon$ for any $x > N_\varepsilon$. As a result, $\mathbb{P}^T(|\mathfrak{R}(X_T^{1,T}) - 1| \ge \varepsilon, X_T^{1,T} > N_\varepsilon) = 0$. Combining this with $\lim_{T\to\infty} \mathbb{P}^T(X_T^{1,T} \le N_\varepsilon) = 0$, the second statement follows. □

The following result is crucial for the proof of Proposition 3.8 later on. Recall that r^T is given in (2.4).

Lemma 3.6

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\big[\big|1 - \mathfrak{R}\big(X_T^{1,T}\big)\big(r_T^T\big)^{p-1}\big|\big|r_T^T - 1\big|\big] = 0.$$

Proof To ease notation, set $\mathfrak{R}_T = \mathfrak{R}(X_T^{1,T})$ and $r_T = r_T^T$. It follows from (3.2) that $\mathbb{E}^{\mathbb{P}}[(X_T^{0,T})^{p-1}(X_T^{1,T} - X_T^{0,T})] \le 0$ and $\mathbb{E}^{\mathbb{P}}[U'(X_T^{1,T})(X_T^{0,T} - X_T^{1,T})] \le 0$. Summing these two inequalities, it follows that

$$\begin{aligned} 0 &\ge \mathbb{E}^{\mathbb{P}}\big[\big((X_T^{0,T})^{p-1} - U'(X_T^{1,T})\big)\big(X_T^{1,T} - X_T^{0,T}\big)\big] \\ &= \mathbb{E}^{\mathbb{P}}\left[\big(X_T^{0,T}\big)^{p-1}\left(1 - \frac{U'(X_T^{1,T})}{(X_T^{1,T})^{p-1}}\frac{(X_T^{1,T})^{p-1}}{(X_T^{0,T})^{p-1}}\right)\big(X_T^{1,T} - X_T^{0,T}\big)\right] \\ &= \mathbb{E}^{\mathbb{P}}\big[\big(X_T^{0,T}\big)^p\big(1 - \mathfrak{R}_T r_T^{p-1}\big)(r_T - 1)\big]. \end{aligned}$$

After changing to the measure $\mathbb{P}^T$, the previous inequality reads

$$\mathbb{E}^{\mathbb{P}^T}\big[\big(1 - \mathfrak{R}_T\, r_T^{p-1}(r_T - 1)\big)\big] \le 0.$$

Note that we have $(1 - \mathfrak{R}_T\, r_T^{p-1})(r_T - 1) \le 0$ if and only if $\mathfrak{R}_T^{1/(1-p)} \le r_T \le 1$ or $1 \le r_T \le \mathfrak{R}_T^{1/(1-p)}$, and so

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^T}\big[\big|1-\mathfrak{R}_T\, r_T^{p-1}\big|\,|r_T-1|\big]\\
&\quad\le 2\,\mathbb{E}^{\mathbb{P}^T}\big[\big(1-\mathfrak{R}_T r_T^{p-1}\big)(1-r_T)\,1_{\{R_T^{1/(1-p)}\le r_T\le 1 \text{ or } 1\le r_T\le R^{1/(1-p)}\}}\big].
\end{aligned}
\tag{3.5}
$$

Let us estimate the right-hand side separately on the sets $\{\mathfrak{R}_T^{1/(1-p)}\le r_T\le 1\}$ and $\{1\le r_T\le \mathfrak{R}_T^{1/(1-p)}\}$. On the first set, note that

$$
\big(1-\mathfrak{R}_T r_T^{p-1}\big)(1-r_T)\le (1-\mathfrak{R}_T)\big(1-\mathfrak{R}_T^{1/(1-p)}\big).
$$

Then

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^T}\big[\big(1-\mathfrak{R}_T r_T^{p-1}\big)(1-r_T)\,1_{\{\mathfrak{R}_T^{1/(1-p)}\le r_T\le 1\}}\big]\\
&\quad\le \mathbb{E}^{\mathbb{P}^T}\big[(1-\mathfrak{R}_T)\big(1-\mathfrak{R}_T^{1/(1-p)}\big)\,1_{\{\mathfrak{R}_T\le 1\}}\big]\\
&\quad\le \mathbb{P}^T(\mathfrak{R}_T\le 1-\epsilon)+\mathbb{E}^{\mathbb{P}^T}\big[(1-\mathfrak{R}_T)\big(1-\mathfrak{R}_T^{1/(1-p)}\big)\,1_{\{1-\epsilon\le\mathfrak{R}_T\le 1\}}\big]\\
&\quad\le \mathbb{P}^T(\mathfrak{R}_T\le 1-\epsilon)+\epsilon\big(1-(1-\epsilon)^{1/(1-p)}\big).
\end{aligned}
$$

Sending $T\to\infty$, then $\epsilon\downarrow 0$, and using Lemma 3.5, it follows that

$$
\lim_{T\to\infty}\mathbb{E}^{\mathbb{P}^T}\big[\big(1-\mathfrak{R}_T r_T^{p-1}\big)(1-r_T)\,1_{\{R_T^{1/(1-p)}\le r_T\le 1\}}\big]=0.
\tag{3.6}
$$

On the set $\{1\le r_T\le \mathfrak{R}_T^{1/(1-p)}\}$, note that $\mathfrak{R}_T r_T^{p-1}+r_T\ge 2$. Then on the same set,

$$
\big(1-\mathfrak{R}_T r_T^{p-1}\big)(1-r_T)=\mathfrak{R}_T r_T^{p}-\mathfrak{R}_T r_T^{p-1}-r_T+1\le \mathfrak{R}_T r_T^{p}-1.
$$

Therefore,

$$
\begin{aligned}
&\mathbb{E}^{\mathbb{P}^T}\big[\big(1-\mathfrak{R}_T r_T^{p-1}\big)(1-r_T)\,1_{\{1\le r_T\le \mathfrak{R}_T^{1/(1-p)}\}}\big]\\
&\quad\le \mathbb{E}^{\mathbb{P}^T}\big[\big(1-\mathfrak{R}_T r_T^{p-1}\big)(1-r_T)\,1_{\{1\le r_T\le \mathfrak{R}_T^{1/(1-p)},\,\mathfrak{R}_T\le 1+\epsilon\}}\big]\\
&\qquad+\mathbb{E}^{\mathbb{P}^T}\big[\big(\mathfrak{R}_T r_T^{p}-1\big)\,1_{\{1\le r_T\le \mathfrak{R}_T^{1/(1-p)},\,1+\epsilon<\mathfrak{R}_T\}}\big]\\
&\quad=: J_1+J_2.
\end{aligned}
$$

In the previous equation, $J_1\le \epsilon((1+\epsilon)^{1/(1-p)}-1)$. Let us focus on J_2 in what follows. Since

$$
J_2\le \mathbb{E}^{\mathbb{P}^T}\big[\big(\mathfrak{R}_T r_T^{p}-1\big)1_{\{1+\epsilon<\mathfrak{R}_T\}}\big]=\mathbb{E}^{\mathbb{P}^T}\big[\mathfrak{R}_T r_T^{p}1_{\{1+\epsilon<\mathfrak{R}_T\}}\big]-\mathbb{P}^T(1+\epsilon<\mathfrak{R}_T),
$$

and $\lim_{T\to\infty}\mathbb{P}^T(1+\epsilon<\mathfrak{R}_T)=0$ from Lemma 3.5, it suffices to estimate the first term in the previous inequality. To this end, note from (2.2) that for some M depending on ϵ, we have $\{1+\epsilon < \mathfrak{R}_T\} \subset \{X_T^{1,T} \le M\}$. Then

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^T}\left[\mathfrak{R}_T r_T^p \, 1_{\{1+\epsilon<\mathfrak{R}_T\}}\right] &\le \mathbb{E}^{\mathbb{P}^T}\left[\mathfrak{R}_T r_T^p \, 1_{\{X_T^{1,T}\le M\}}\right] \\ &= \frac{\mathbb{E}^{\mathbb{P}}[U'(X_T^{1,T})X_T^{1,T} \, 1_{\{X_T^{1,T}\le M\}}]}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,T})^p]} \\ &= \frac{y^{1,T}}{y^{0,T}}\mathbb{E}^{\mathbb{P}}\left[Y_T^{1,T}X_T^{1,T} \, 1_{\{X_T^{1,T}\le M\}}\right].\end{aligned}$$

Introduce the probability measure $\mathbb{P}^{1,T}$ via

$$\frac{d\mathbb{P}^{1,T}}{d\mathbb{P}} := Y_T^{1,T} X_T^{1,T}.$$

A line of reasoning similar to that in Remark 3.2(iii) shows that $X^{1,T}$ has the numéraire property under $\mathbb{P}^{1,T}$. Thus, the argument in Lemma 3.3 applied to $X^{1,T}$ and $\mathbb{P}^{1,T}$ implies that $\lim_{T\to\infty}\mathbb{P}^{1,T}(X^{1,T}\ge M)=1$. The previous convergence, combined with Lemma 3.4, then implies

$$\frac{y^{1,T}}{y^{0,T}}\mathbb{E}^{\mathbb{P}}\left[Y_T^{1,T}X_T^{1,T} \, 1_{\{X_T^{1,T}\le M\}}\right] = \frac{y^{1,T}}{y^{0,T}}\mathbb{P}^{1,T}\left(X_T^{1,T}\le M\right) \longrightarrow 0 \quad \text{as } T\to\infty.$$

Now, combining the estimates on J_1 and J_2 and utilizing Lemma 3.5, sending $T\to\infty$ and then $\epsilon \downarrow 0$, it follows that

$$\lim_{T\to\infty}\mathbb{E}^{\mathbb{P}^T}\left[\left(1-\mathfrak{R}_T r_T^{p-1}\right)(1-r_T) \, 1_{\{1\le r_T\le \mathfrak{R}_T^{1/(1-p)}\}}\right]=0.$$

Combining this with (3.6), the statement now follows from (3.5). □

The previous result implies that $r_T^T \to 1$ under $\mathbb{P}^T$.

Lemma 3.7

$$\lim_{T\to\infty}\mathbb{P}^T\left(\left|r_T^T-1\right|\ge\epsilon\right)=0, \quad \textit{for any } \epsilon>0.$$

Proof As in the proof of Lemma 3.6, set $\mathfrak{R}_T=\mathfrak{R}(X_T^{1,T})$ and $r_T=r_T^T$. Fix $\epsilon\in(0,1)$ and consider the set

$$D^T=\left\{|r_T-1|\ge\epsilon, (1-\epsilon)^{\frac{1-p}{2}}\le\mathfrak{R}_T\le(1+\epsilon)^{\frac{1-p}{2}}\right\}.$$

The next task is to estimate the lower bound of $|1-\mathfrak{R}_T r_T^{p-1}|$ on D^T for the cases $r_T\ge 1+\epsilon$ and $r_T\le 1-\epsilon$ separately.

For $r_T\ge 1+\epsilon$, we have $\mathfrak{R}_T r_T^{p-1}\le(1+\epsilon)^{\frac{p-1}{2}}<1$ on D^T, whence

$$1-\mathfrak{R}_T r_T^{p-1}\ge 1-(1+\epsilon)^{\frac{p-1}{2}}>0.$$

For $r_T \le 1-\epsilon$, we have $\mathfrak{R}_T r_T^{p-1} \ge (1-\epsilon)^{\frac{p-1}{2}} > 1$ on D^T, whence

$$1-\mathfrak{R}_T r_T^{p-1} \le 1-(1-\epsilon)^{\frac{p-1}{2}} < 0.$$

Denote $\eta = \min\{1-(1+\epsilon)^{\frac{p-1}{2}}, -1+(1-\epsilon)^{\frac{p-1}{2}}\}$. In either of the above cases, $|1-\mathfrak{R}_T r_T^{p-1}| \ge \eta$; therefore,

$$\mathbb{E}^{\mathbb{P}^T}\left[\left|1-\mathfrak{R}_T r_T^{p-1}\right| |r_T - 1|\right] \ge \epsilon\, \eta\, \mathbb{P}^T\left(D^T\right).$$

Combining this with Lemma 3.6, it follows that

$$\lim_{T\to\infty} \mathbb{P}^T\left(D^T\right) = 0.$$

From this and the second statement in Lemma 3.5, the proof is complete. □

The previous results allow to prove the main result of this subsection.

Proposition 3.8

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[\left|r_T^T - 1\right|\right] = 0.$$

Proof As in the previous lemmas, set $r_T = r_T^T$. The proof consists of the following two steps, whose combination confirms the claim. Note that for $p=0$, $\mathbb{P}^T$ below is exactly $\mathbb{P}$ and hence convergence takes place under the physical measure.

Step 1: Establishing that

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[|r_T - 1| 1_{\{r_T \le N\}}\right] = 0, \quad \text{for any } N > 2. \tag{3.7}$$

To this end, note that for any $\epsilon > 0$,

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}^T}\left[|r_T - 1| 1_{\{r_T \le N\}}\right] \\
&\quad = \mathbb{E}^{\mathbb{P}^T}\left[|r_T - 1| 1_{\{r_T \le N,\, |r_T - 1| \le \epsilon\}}\right] + \mathbb{E}^{\mathbb{P}^T}\left[|r_T - 1| 1_{\{r_T \le N,\, |r_T - 1| > \epsilon\}}\right] \\
&\quad \le \epsilon + (N-1)\, \mathbb{P}^T\left(|r_T - 1| > \epsilon\right).
\end{aligned}$$

As $T \to \infty$, (3.7) follows from Lemma 3.7 and the arbitrary choice of ϵ.

Step 2: Showing that

$$\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[|r_T - 1| 1_{\{r_T > N\}}\right] = 0, \quad \text{for any } N > 2. \tag{3.8}$$

To this end,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}^T}\left[|r_T - 1| 1_{\{r_T > N\}}\right] &\le \mathbb{E}^{\mathbb{P}^T}\left[r_T 1_{\{r_T > N\}}\right] \\
&= \mathbb{E}^{\mathbb{P}^T}\left[r_T\right] - \mathbb{E}^{\mathbb{P}^T}\left[(r_T - 1) 1_{\{r_T \le N\}}\right] - \mathbb{P}^T(r_T \le N).
\end{aligned}$$

Note that $\mathbb{E}^{\mathbb{P}^T}[r_T] \leq 1$ by the numéraire property of $X^{0,T}$ under $\mathbb{P}^T$ (cf. Remark 3.2(iii)), $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}[(r_T - 1)1_{\{r_T \leq N\}}] = 0$ from Step 1, and $\lim_{T\to\infty} \mathbb{P}^T(r_T \leq N) = 1$ from Lemma 3.7. Therefore,

$$0 \leq \limsup_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}\left[|r_T - 1| 1_{\{r_T > N\}}\right] \leq 1 - 0 - 1 = 0,$$

which confirms (3.8). □

3.2 Convergence of wealth processes

The following lemma bridges the transition from the convergence of optimal payoffs to the convergence of their wealth processes.

Lemma 3.9 *Consider a sequence $(r^T)_{T\in\mathbb{R}_+}$ of càdlàg processes and a sequence $(\mathbb{P}^T)_{T\in\mathbb{R}_+}$ of probability measures such that*

(i) *for each $T \in \mathbb{R}_+$, r^T is defined on $[0, T]$ with $r_0^T = 1$ and $r_t^T > 0$ for all $t \leq T$, $\mathbb{P}^T$-a.s.;*
(ii) *each r^T is a $\mathbb{P}^T$-supermartingale on $[0, T]$;*
(iii) $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}[|r_T^T - 1|] = 0$.

Then:

(a) $\lim_{T\to\infty} \mathbb{P}^T(\sup_{u\in[0,T]} |r_u^T - 1| \geq \epsilon) = 0$, *for any $\epsilon > 0$;*
(b) *define $L^T := \int_0^\cdot (1/r_{t-}^T)\, dr_t^T$, i.e., L^T is the stochastic logarithm of r^T, for each $T \in \mathbb{R}_+$. Then $\lim_{T\to\infty} \mathbb{P}^T([L^T, L^T]_T \geq \epsilon) = 0$, for any $\epsilon > 0$, where $[\cdot,\cdot]_T$ is the square bracket on $[0, T]$.*

Proof This follows from Theorem 2.5 and Remark 2.6 in Kardaras [34]. (Note that Theorem 2.5 in [34] is stated under a fixed probability $\overline{\mathbb{P}}$ and on a fixed time interval $[0, T]$, but its proof remains valid for a sequence of probability measures $(\mathbb{P}^T)_{T\in\mathbb{R}_+}$ and a family of time intervals $([0, T])_{T\in\mathbb{R}_+}$.) □

Combining Lemma 3.9 with Proposition 3.8, Proposition 2.5 is proved as follows.

Proof of Proposition 2.5 The statements follow directly from Lemma 3.9, after checking that its assumptions are satisfied. First, $r_0^T = 1$ since both investors have the same initial capital. Second, assuming r^T to be a $\mathbb{P}^T$-supermartingale for a moment, we get $r_t^T > 0$ $\mathbb{P}^T$-a.s. for any $t \leq T$, because $r_T^T > 0$ $\mathbb{P}^T$-a.s. (see Remark 3.2(i)). Third, $\lim_{T\to\infty} \mathbb{E}^{\mathbb{P}^T}[|r_T^T - 1|] = 0$ is the result of Proposition 3.8. Hence it remains to show that r^T is a $\mathbb{P}^T$-supermartingale. To this end, it suffices to show that

$$\mathbb{E}^{\mathbb{P}^T}\left[X_t / X_t^{0,T} \middle| \mathcal{F}_s\right] \leq X_s / X_s^{0,T}, \quad \text{for any } s < t \leq T \text{ and } X \in \mathcal{X}. \tag{3.9}$$

Since $X_T^{0,T} > 0$ $\mathbb{P}^T$-a.s., Remark 3.2(i) implies that both denominators in (3.9) are nonzero. To prove (3.9), fix any $A \in \mathcal{F}_s$ and construct the wealth process $\widetilde{X} \in \mathcal{X}$ via

$$\widetilde{X}_u := \begin{cases} X_u^{0,T}, & u \in [0,s), \\ X_s^{0,T} \frac{X_u}{X_s} 1_A + X_u^{0,T} 1_{\Omega \setminus A}, & u \in [s,t), \\ X_s^{0,T} \frac{X_t}{X_s} \frac{X_u^{0,T}}{X_t^{0,T}} 1_A + X_u^{0,T} 1_{\Omega \setminus A}, & u \in [t,T]. \end{cases}$$

Noting that

$$\frac{\widetilde{X}_T}{X_T^{0,T}} = \frac{X_s^{0,T}}{X_s} \frac{X_t}{X_t^{0,T}} 1_A + 1_{\Omega \setminus A},$$

the claim follows from $\mathbb{E}^{\mathbb{P}^T}[\widetilde{X}_T / X_T^{0,T}] \le 1$ (cf. Remark 3.2(iii)) and the arbitrary choice of A. □

Proof of Lemma 2.6 Set $X^T = d\mathbb{Q}^T / d\widetilde{\mathbb{Q}}|_{\mathcal{F}_t}$. Note that for all $0 < \delta < 1$ and the given $\varepsilon > 0$,

$$\begin{aligned} \widetilde{\mathbb{Q}}[A_T] = \widetilde{\mathbb{Q}}[1_{A_T} \ge \delta] &= \widetilde{\mathbb{Q}}[1_{A_T} \ge \delta, X^T \ge \varepsilon] + \widetilde{\mathbb{Q}}[1_{A_T} \ge \delta, X^T < \varepsilon] \\ &\le \widetilde{\mathbb{Q}}[X^T 1_{A_T} \ge \varepsilon\delta] + \widetilde{\mathbb{Q}}[X^T < \varepsilon]. \end{aligned}$$

Since $\lim_{T\to\infty} \mathbb{Q}^T[A_T] = 0$, it follows that $X^T 1_{A_T}$ goes to 0 in $\widetilde{\mathbb{Q}}$-probability. By hypothesis, $\lim_{T\to\infty} \widetilde{\mathbb{Q}}[X^T < \varepsilon] = 0$. Therefore, $\lim_{T\to\infty} \widetilde{\mathbb{Q}}[A_T] = 0$ holds. The statement then follows from the equivalence between $\mathbb{Q}$ and $\widetilde{\mathbb{Q}}$. □

Proof of Corollary 2.7 First, note that $(d\mathbb{P}^T / d\mathbb{P}|_{\mathcal{F}_t})_{T \ge t}$ is a constant sequence. Indeed, for any $t \le T \le S$,

$$\begin{aligned} \frac{d\mathbb{P}^S}{d\mathbb{P}}\bigg|_{\mathcal{F}_t} &= \frac{\mathbb{E}_t^{\mathbb{P}}[(X_S^{0,S})^p]}{\mathbb{E}^{\mathbb{P}}[(X_S^{0,S})^p]} = \frac{\mathbb{E}_t^{\mathbb{P}}[(X_T^{0,S})^p (X_S^{0,S}/X_T^{0,S})^p]}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,S})^p (X_S^{0,S}/X_T^{0,S})^p]} \\ &= \frac{\mathbb{E}_t^{\mathbb{P}}[(X_T^{0,S})^p]\mathbb{E}_t^{\mathbb{P}}[(X_S^{0,S}/X_T^{0,S})^p]}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,S})^p]\mathbb{E}^{\mathbb{P}}[(X_S^{0,S}/X_T^{0,S})^p]} \\ &= \frac{\mathbb{E}_t^{\mathbb{P}}[(X_T^{0,S})^p]}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,S})^p]} = \frac{\mathbb{E}_t^{\mathbb{P}}[(X_T^{0,T})^p]}{\mathbb{E}^{\mathbb{P}}[(X_T^{0,T})^p]} = \frac{d\mathbb{P}^T}{d\mathbb{P}}\bigg|_{\mathcal{F}_t}. \end{aligned}$$

Here, the third equality follows from the assumption that $X_T^{0,S}$ and $X_S^{0,S}/X_T^{0,S}$ are independent; the fourth equality holds since $X_S^{0,S}/X_T^{0,S}$ is independent of $\mathcal{F}_t$; and the fifth equality holds by the myopic optimality $X_T^{0,T} = X_T^{0,S}$.

It follows from the last paragraph that $d\mathbb{P}^T/d\mathbb{P}|_{\mathcal{F}_t} = d\mathbb{P}^t/d\mathbb{P}|_{\mathcal{F}_t}$ $\mathbb{P}$-a.s. for any $T \ge t$. Then the statement follows since $\mathbb{P}^t$ is equivalent to $\mathbb{P}$ on $\mathcal{F}_t$; see the discussion after (2.5). □

4 Proof of the turnpike for diffusions

4.1 Outline of the proof

This section contains the proofs of the results in Sects. 2.3.4 and 2.3.5. The discussion starts in Sect. 4.2 with the construction of the long-run measures $(\hat{\mathbb{P}}^{\xi})_{\xi\in\mathbb{R}^d\times E}$ and their properties. First $(\hat{\mathbb{P}}^y_Y)_{y\in E}$, the restriction of $(\hat{\mathbb{P}}^{\xi})_{\xi\in\mathbb{R}^d\times E}$ to the last component of the state space, is constructed in Lemma 4.1. Assumption 2.13 implies that Y is positive recurrent under $(\hat{\mathbb{P}}^y_Y)_{y\in E}$ with invariant density $\hat{v}^2(y)\hat{m}(y)$. The long-run limit of $\mathbb{E}^{\hat{\mathbb{P}}^y_Y}[f(Y_T)]$ is then established for functions f integrable with respect to the invariant density. This property is used to study the long-run limit of $d\mathbb{P}^{T,y}/d\hat{\mathbb{P}}^y|_{\mathcal{F}_t}$ in Lemma 2.17. In Lemma 4.2, $(\hat{\mathbb{P}}^{\xi})_{\xi\in\mathbb{R}^d\times E}$ is constructed by adding the first d components to $(\hat{\mathbb{P}}^y_Y)_{y\in E}$, and each $\hat{\mathbb{P}}^{\xi}$ is shown to be equivalent to the physical measure $\mathbb{P}^{\xi}$.

Section 4.3 is devoted to the construction of the candidate reduced value function v^T and the verification that it is indeed the reduced value function. To this end, we first consider h^T, which is expected to solve (2.21). Instead of starting from (2.21) and showing that its solution admits the stochastic representation (2.23), we define h^T via the stochastic representation and verify via a localization argument that it is a classical solution to (2.21). This approach avoids both uniform ellipticity and growth assumptions on the terminal condition which come with the classic version of the Feynman–Kac formula. After the reduced value function is constructed in (4.3), its relationship with the value function u^T is verified in Proposition 4.6.

Section 4.4 establishes the precise relations between the densities $d\mathbb{P}^{T,y}/d\mathbb{P}^y|_{\mathcal{F}_t}$, $d\hat{\mathbb{P}}^y/d\mathbb{P}^y|_{\mathcal{F}_t}$ and the wealth processes $X^{0,T}$ and $\hat{X}$. These relations prepare the proofs of the main results in the last subsection.

4.2 The long-run measure $\hat{\mathbb{P}}^{\xi}$

Recall the following terminology from ergodic theory for diffusions (see Pinsky [46] for a more thorough treatment). Let $\check{L}$ be as in (2.9). Suppose the martingale problem for $\check{L}$ is well posed on D, and denote its solution by $(\mathbb{P}^x)_{x\in D}$, with coordinate process Ξ. Denote by $\check{L}^*$ the formal adjoint to $\check{L}$. Note that under Assumption 2.8, $\check{L}^*$ is a second-order differential operator.

Ξ is *recurrent* under $(\mathbb{P}^x)_{x\in D}$ if $\mathbb{P}^x(\tau(\epsilon, y) < \infty) = 1$ for any $(x, y) \in D^2$ and $\epsilon > 0$, where $\tau(\epsilon, y) = \inf\{t \geq 0 \,|\, |\Xi_t - y| < \epsilon\}$. If Ξ is recurrent, then it is *positive recurrent* or *ergodic* if there exists a strictly positive $\varphi^* \in C^{2,\gamma}(D, \mathbb{R}^k)$ such that $\check{L}^*\varphi^* = 0$ and $\int_D \varphi^*(y)\,dy < \infty$. If Ξ is recurrent but not positive recurrent, it is *null recurrent*; cf. [46, Chap. 4] for more details.

Lemma 4.1 *Let Assumptions 2.8 and 2.13 hold. Let $\mathcal{L}$ be as in (2.14) and $\mathcal{L}^{\hat{v},0}$ as in (2.22). Then there exists a unique solution $(\hat{\mathbb{P}}^y_Y)_{y\in E}$ to the martingale problem for $\mathcal{L}^{\hat{v},0}$ on E. Furthermore, the coordinate process Y is positive recurrent under $(\hat{\mathbb{P}}^y_Y)_{y\in E}$ with invariant density $\hat{v}^2(y)\hat{m}(y)$, where $\hat{m}$ is defined in (2.26). Therefore, for all functions f integrable with respect to the invariant density and all $t > 0$,*

$$\lim_{T\to\infty} \mathbb{E}^{\hat{\mathbb{P}}^y_Y}[f(Y_{T-t})] = \int_E f(y)\hat{v}^2(y)\hat{m}(y)\,dy. \tag{4.1}$$

Proof Since (2.24) and (2.25) hold, applying [46, Theorem 5.1.10, Corollary 5.1.11] to the operator $\mathcal{L}^{\hat{v},0}$, whose drift term is $B + A\hat{v}_y/\hat{v}$, yields that $(\hat{\mathbb{P}}_Y^y)_{y\in E}$ exists and is unique, Y is positive recurrent under $(\hat{\mathbb{P}}_Y^y)_{y\in E}$, and Y has the invariant density $\hat{v}^2(y)\hat{m}(y)$. That (4.1) holds for f integrable with respect to the invariant density follows from [44, Theorem 1.2(iii), Eqs. (3.29) and (3.30)] or [45, Corollary 5.2]. □

Lemma 4.2 *Let Assumptions 2.8 and 2.13 hold. Then*:

(i) *There exists a unique solution $(\hat{\mathbb{P}}^\xi)_{\xi\in\mathbb{R}^d\times E}$ to the martingale problem for $\hat{\mathcal{L}}$ on $\mathbb{R}^d \times E$, where $\hat{\mathcal{L}}$ is given in (2.29)*;
(ii) *$\hat{\mathbb{P}}^\xi \sim \mathbb{P}^\xi$ on $\mathcal{F}_t$, for any $t \geq 0$ and $\xi \in \mathbb{R}^d \times E$.*

Proof For any integer n, denote by Ω^n the space of continuous maps $\omega : \mathbb{R}_+ \to \mathbb{R}^n$ and by $\mathcal{B}^n$ the σ-algebra generated by the coordinate process $\Xi = (\Xi^1, \dots, \Xi^{d+1})$ defined by $\Xi_t(\omega) = \omega_t$ for $\omega \in \Omega^n$. By Lemma 4.1, there is a unique solution $(\hat{\mathbb{P}}_Y^y)_{y\in E}$ on $(\Omega^1, \mathcal{B}^1)$ to the martingale problem on E for the operator $\mathcal{L}^{\hat{v},0}$ given in (2.22). Set $\Omega = \Omega^{d+1}$, $\mathcal{F} = \mathcal{B}^{d+1}$ and $\mathcal{F}_t = \mathcal{B}^{d+1}_{t+}, t \geq 0$. Let $\mathcal{W}^d$ denote d-dimensional Wiener measure on the first d coordinates (along with the associated σ-algebra) and set $\hat{B} = (\Xi^1, \dots, \Xi^d)$, $Y = \Xi^{d+1}$. For any $z \in \mathbb{R}^d$, define the processes $\hat{W}$, R by

$$\hat{W}_t = \int_0^t a^{-1}(Y_s)\big(dY_s - b(Y_s)\,ds\big),$$
$$R_t = z + \int_0^t \frac{1}{1-p}\left(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}}\right)(Y_s)\,ds + \int_0^t \sigma(Y_s)\rho\,d\hat{W}_s + \int_0^t \sigma(Y_s)\overline{\rho}\,d\hat{B}_s.$$

For $\xi = (z, y)$, it follows that $((R, Y), (\hat{B}, \hat{W}), (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathcal{W}^d \times \hat{\mathbb{P}}_Y^y))$ is a weak solution to the SDE

$$\begin{aligned} dR_t &= \frac{1}{1-p}\left(\mu + \delta\Upsilon\frac{\hat{v}_y}{\hat{v}}\right)(Y_t)\,dt + \sigma(Y_t)(\rho\,d\hat{W}_t + \bar{\rho}\,d\hat{B}_t), \\ dY_t &= \left(B + A\frac{\hat{v}_y}{\hat{v}}\right)(Y_t)\,dt + a(Y_t)\,d\hat{W}_t. \end{aligned} \tag{4.2}$$

Since weak solutions induce solutions to the martingale problem via Itô's formula, $\hat{\mathbb{P}}^\xi$ defined as the law of (R, Y) solves the martingale problem for $\hat{\mathcal{L}}$. It is also the unique solution, since Assumption 2.8 and $\hat{v} \in C^2(E)$ in Assumption 2.13 imply that the coefficients of (4.2) are locally Lipschitz; hence uniqueness in law holds for (4.2).

Part (ii) follows from [6, Remark 2.6]. Note that the assumption in [6] is satisfied in view of Assumption 2.8, $\hat{v} > 0$ and $\hat{v} \in C^2(E)$ in Assumption 2.13. □

Remark 4.3 As in the proof of Lemma 4.2, for all $\xi = (z, y)$ with $z \in \mathbb{R}^d$ and $y \in E$, if Y denotes the $(d+1)$th coordinate, then

$$\hat{\mathbb{P}}^\xi(Y \in A) = \hat{\mathbb{P}}_Y^y(Y \in A), \quad A \in \mathcal{B}^1.$$

Thus, since Y is positive recurrent under $(\hat{\mathbb{P}}^y_Y)_{y\in E}$ by Lemma 4.1, Y is positive recurrent under $(\hat{\mathbb{P}}^\xi)_{\xi\in\mathbb{R}^d\times E}$ with the same invariant density as in Lemma 4.1. Therefore, the ergodic result in (4.1) applies to $\hat{\mathbb{P}}^\xi$ for any $\xi\in\mathbb{R}^d\times E$.

4.3 Construction of v^T

The solution $v^T(t,y)$ to (2.13) is constructed from the long-run solution $\hat{v}(y)$ of Assumption 2.13. Recall that $\hat{\mathbb{P}}^\xi$ is denoted by $\hat{\mathbb{P}}^y$ for $\xi=(0,y)$. Now consider the function h^T defined by (2.23). The candidate reduced value function is

$$v^T(t,y) := e^{\lambda_c(T-t)}\hat{v}(y)h^T(t,y). \tag{4.3}$$

The verification result in Proposition 4.6 below confirms that v^T is a strictly positive classical solution to (2.13), and the relation $u^T(t,x,y)=(x^p/p)(v^T(t,y))^\delta$ holds for $(t,x,y)\in[0,T]\times\mathbb{R}_+\times E$.

As a first step to proving Proposition 4.6, the next result characterizes the function h^T. Clearly, $h^T(t,y)>0$ for $(t,y)\in[0,T]\times E$.

Proposition 4.4 *Let Assumptions 2.8, 2.9 and 2.13 hold. Then $h^T(t,y)<\infty$ for all $(t,y)\in[0,T]\times E$, $h^T(t,y)\in C^{1,2}((0,T)\times E)$, and h^T satisfies (2.21). Moreover, the process*

$$\frac{h^T(t,Y_t)}{h^T(0,y)},\quad 0\le t\le T, \tag{4.4}$$

is a $\hat{\mathbb{P}}^y$-martingale on $[0,T]$ with constant expectation 1. Furthermore, for all $t>0$ and $y\in E$, it follows $\hat{\mathbb{P}}^y$-almost surely that

$$\lim_{T\to\infty}\frac{h^T(t,Y_t)}{h^T(0,y)}=1. \tag{4.5}$$

Proof The proof consists of several steps.

Step 1: $h^T(t,y)<\infty$ for all $(t,y)\in[0,T]\times E$: Note that for $t=T$, we have $h^T(t,y)=1/\hat{v}(y)<\infty$ by definition. Now fix $t<T$ and $y\in E$. Using the transition density $\hat{p}$ for Y under $(\hat{\mathbb{P}}^y)_{y\in E}$, we get

$$h^T(t,y)=\int_E \hat{p}(y,T-t,z)\frac{1}{\hat{v}(z)}\,dz.$$

Since $T-t>0$, according to [44, (3.30)], for each $y\in E$ there is some constant $C(T-t,y)$ such that $\hat{p}(y,T-t,z)\le C(T-t,y)\hat{v}^2(z)\hat{m}(z)$, where $\hat{v}^2\hat{m}$ is the invariant density for Y under $(\hat{\mathbb{P}}^y)_{y\in E}$. Thus,

$$h^T(t,y)\le C(T-t,y)\int_E\left(\hat{v}^2\hat{m}\frac{1}{\hat{v}}\right)(z)\,dz = C(T-t,y)\int_E \hat{v}(z)\hat{m}(z)\,dz<\infty, \tag{4.6}$$

where the last inequality follows from (2.25).

Remark 4.5 As shown in [44], for $t < T - 1$ ($T > 1$), the constant $C(T-t, y)$ can be made uniform in T.

Step 2: $h^T \in C^{1,2}((0,T) \times E)$ satisfies (2.21): For this purpose, the classic version of the Feynman–Kac formula (see Theorem 5.3 in [19, Chap. 6]) does not apply directly because the operator $\mathcal{L}^{\hat{v},0}$ is not assumed to be uniformly elliptic on E and $1/\hat{v}$ may grow faster than polynomially near the boundary of E. Rather, the statement follows from Theorem 1 in Heath and Schweizer [24], which yields that h^T is a classical solution of (2.21). To check that the assumptions of Theorem 1 in [24] are satisfied, first note that clearly, since $\hat{v} \in C^2(E)$, the given assumptions imply that Assumptions (A1), (A2) and (A3a′)–(A3d′) are satisfied.

To check (A3e′), it suffices to show that h^T is continuous in any compact subdomain of $(0,T) \times E$. Recall that the domain is $E = (\alpha, \beta)$ for $-\infty \le \alpha < \beta \le \infty$. Let (α_m) and (β_m) be two sequences such that $\alpha_m < \beta_m$ for all m, α_m strictly decreases to α and β_m strictly increases to β. Set $E_m = (\alpha_m, \beta_m)$. For each m, there exists a function $\psi_m(y) \in C^\infty(E)$ such that $\psi_m(y) \le 1$, $\psi_m(y) = 1$ on E_m, and $\psi_m(y) = 0$ on $E \cap E^c_{m+1}$. To construct such ψ_m, let $\varepsilon_m = (1/3)\min(\beta_{m+1} - \beta_m, \alpha_m - \alpha_{m+1})$ and set $\psi_m(y) := \eta_{\varepsilon_m} * 1_{(\alpha_m - \varepsilon_m, \beta_m + \varepsilon_m)}(y)$, where η_{ϵ_m} is the standard mollifier on $(-\epsilon_m, \epsilon_m)$ for some $\epsilon > 0$ and $*$ is the convolution operator; cf. [13, Appendix C.4]. Define the functions f_m and $h^{T,m}$ by

$$f_m(y) := \frac{\psi_m(y)}{\hat{v}(y)} \quad \text{and} \quad h^{T,m}(t,y) := \mathbb{E}^{\hat{P}^y}[f_m(Y_{T-t})].$$

By construction, for all $y \in E$, $f_m(y)$ strictly increases to $1/\hat{v}(y)$. It then follows from monotone convergence and (4.6) that $\lim_{m\to\infty} h^{T,m}(t,y) = h^T(t,y)$. Since $\hat{v} \in C^2(E)$ and $\hat{v} > 0$, each $f_m \in C^2(E)$ is bounded, so that the Feller property for $\hat{\mathbb{P}}^y$ (see Theorem 1.13.1 in [46]) implies that $h^{T,m}$ is continuous in y. On the other hand, by the construction of f_m and the fact that $\mathcal{L}^{\hat{v},0}(1/\hat{v}) = (c - \lambda_c)(1/\hat{v})$, there exists a constant $K_m > 0$ such that

$$a|\dot{f}_m| + \left|\mathcal{L}^{\hat{v},0} f_m\right| \le K_m \quad \text{on } E.$$

Therefore, for $0 \le s \le t \le T$, Itô's formula implies that

$$\sup_{y\in E}\left|\mathbb{E}^{\hat{\mathbb{P}}^y}[f_m(Y_t) - f_m(Y_s)]\right| \le K_m(t-s),$$

and hence $h^{T,m}$ is uniformly continuous in t. Together with the continuity of $h^{T,m}$ in y, it follows that $h^{T,m}$ is jointly continuous in (t,y) on $[0,T] \times E$.

Now note that the operator $\mathcal{L}^{\hat{v},0}$ is uniformly elliptic in the parabolic domain $(0,T) \times E_m$. It then follows from a straightforward calculation that $h^{T,m}$ satisfies the partial differential equation

$$\partial_t h^{T,m} + \mathcal{L}^{\hat{v},0} h^{T,m} = 0, \quad (t,y) \in (0,T) \times E_m.$$

Note that $(h^{T,m})_{m\in\mathbb{N}}$ is uniformly bounded from above by h^T, which is finite on $[0,T]\times E_m$. Appealing to the interior Schauder estimate (see e.g. Theorem 15 in Friedman [18, Chap. 3]), there exists a subsequence $(h^{T,m'})_{m'\in\mathbb{N}}$ which converges to h^T uniformly in $(0,T)\times D$ for any compact sub-domain D of E_m. Since each $h^{T,m'}$ is continuous and the convergence is uniform, h^T is indeed continuous in $(0,T)\times D$. Since the choice of D is arbitrary in E_m, (A3e′) in [24] is satisfied. This proves that h^T is in $C^{1,2}((0,T)\times E)$ and satisfies (2.21).

Step 3: Remaining statements: By the definition of the martingale problem, the process in (4.4) is a nonnegative local martingale, and hence a supermartingale. Furthermore, for $y\in E$, by construction of h^T,

$$\mathbb{E}^{\hat{\mathbb{P}}^y}\left[\frac{h^T(T,Y_T)}{h^T(0,y)}\right]=\frac{1}{\mathbb{E}^{\hat{\mathbb{P}}^y}[1/\hat{v}(Y_T)]}\mathbb{E}^{\hat{\mathbb{P}}^y}[1/\hat{v}(Y_T)]=1,$$

proving the martingale property on $[0,T]$. Lastly, (4.5) follows from (4.1) in Lemma 4.1 and Remark 4.3, since (2.25) in Assumption 2.13 implies that $1/\hat{v}\in L^1(E,\hat{v}^2\hat{m})$. Thus, for all $t>0$, $y\in E$,

$$\lim_{T\to\infty} h^T(t,y)=\int_E \hat{v}(z)\hat{m}(z)\,dz, \tag{4.7}$$

which gives the result by taking $y=Y_t$ for a fixed t. □

The next step towards the verification result in Proposition 4.6 is to connect solutions v^T to the PDE in (2.13) to the value function u^T of (2.11).

Proposition 4.6 *Let Assumptions 2.8, 2.9, and 2.13 hold. Define v^T by (4.3). Then*:

(i) $v^T>0$, $v^T\in C^{1,2}((0,T)\times E)$, *and* v^T *solves* (2.13);
(ii) $u^T(t,x,y)=\frac{x^p}{p}(v^T(t,y))^\delta$ *on* $[0,T]\times\mathbb{R}_+\times E$, *and* π^T *in* (2.15) *is the optimal portfolio.*

Proof Clearly, the positivity of h^T and $\hat{v}$ yield that of v^T. Furthermore, given that h^T solves (2.21), long but straightforward calculations using (2.17) show that v^T solves (2.13). Moreover, v^T is in $C^{1,2}((0,T)\times E)$ because $\hat{v}$ is in $C^2(E)$ and h^T is in $C^{1,2}((0,T)\times E)$. This proves (i).

As for part (ii), by Lemma A.3 in Guasoni and Robertson [22], it suffices to show that for all $y\in E$, the process D^{v^T} from (2.16) is a $\mathbb{P}^y$-martingale on $[0,T]$, or equivalently, that $1=\mathbb{E}^{\mathbb{P}^y}[D_T^{v^T}]$. Note that Lemma A.3 in [22] also proves that if v^T solves (2.13), then for π as in (2.15), $D_t^{v^T}$ satisfies the equality in (2.16). It follows from (4.3) that

$$\frac{v_y^T}{v^T}=\frac{\hat{v}_y}{\hat{v}}+\frac{h_y^T}{h^T}. \tag{4.8}$$

Equation (4.8) and the $\mathbb{P}^y$-independence of Y and B (see Remark 2.11) imply [32, Lemma 4.8]:

$$\begin{aligned}\mathbb{E}^{\mathbb{P}^y}\big[D_T^{v^T}\big] &= \mathbb{E}^{\mathbb{P}^y}\left[\mathcal{E}\left(\int\left(-q\Upsilon'\Sigma^{-1}\mu + A\left(\frac{\hat{v}_y}{\hat{v}} + \frac{h_y^T}{h^T}\right)\right)'\frac{1}{a}\,dW\right)_T\right]\\ &= \mathbb{E}^{\mathbb{P}^y}\left[\mathcal{E}\left(\int\left(-q\Upsilon'\Sigma^{-1}\mu + A\left(\frac{\hat{v}_y}{\hat{v}} + \frac{h_y^T}{h^T}\right)\right)'\frac{1}{a}\,dW\right.\right.\\ &\qquad \left.\left. - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{\hat{v}_y}{\hat{v}}\right)'\sigma\bar{\rho}\,dB\right)_T\right]. \end{aligned}\tag{4.9}$$

Let $\hat{v}$ be as in Assumption 2.13 and recall the definition of $D^{\hat{v}}$ in (2.19) as

$$\begin{aligned}D_t^{\hat{v}} = \mathcal{E}\left(\int\left(-q\Upsilon'\Sigma^{-1}\mu + A\frac{\hat{v}_y}{\hat{v}}\right)'\frac{1}{a}\,dW\right.\\ \left. - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{\hat{v}_y}{\hat{v}}\right)'\sigma\bar{\rho}\,dB\right)_t.\end{aligned}$$

As shown in Guasoni and Robertson [21, Theorem 7], $D^{\hat{v}}$ is a strictly positive $(\mathbb{P}^y, (\mathcal{B}_t)_{t\geq 0})$-martingale; hence for any $t \geq 0$ and $y \in E$, $d\hat{\mathbb{P}}^y/d\mathbb{P}^y|_{\mathcal{B}_t} = D_t^{\hat{v}}$. It follows from the backward martingale convergence theorem that $D^{\hat{v}}$ is a $(\mathbb{P}^y, (\mathcal{B}_{t+})_{t\geq 0})$-martingale. This property still holds after adding all N-negligible sets to $(\mathcal{B}_{t+})_{t\geq 0}$, whence

$$\left.\frac{d\hat{\mathbb{P}}^y}{d\mathbb{P}^y}\right|_{\mathcal{F}_t} = D_t^{\hat{v}}.\tag{4.10}$$

Furthermore, the Brownian motion $\hat{W}$ from (4.2) is related to W by the equality $d\hat{W}_t = dW_t + (q\rho\sigma'\Sigma^{-1}\mu - a\hat{v}_y/\hat{v})\,dt$. Using this, for all $t \leq T$,

$$\begin{aligned}&\mathcal{E}\left(\int\left(-q\Upsilon'\Sigma^{-1}\mu + A\left(\frac{\hat{v}_y}{\hat{v}} + \frac{h_y^T}{h^T}\right)\right)'\frac{1}{a}\,dW\right.\\ &\qquad\left. - q\int\left(\Sigma^{-1}\mu + \Sigma^{-1}\Upsilon\delta\frac{\hat{v}_y}{\hat{v}}\right)'\sigma\bar{\rho}\,dB\right)_t\\ &= D_t^{\hat{v}}\mathcal{E}\left(\int a\frac{h_y^T}{h^T}\,d\hat{W}\right)_t = D_t^{\hat{v}}\frac{h^T(t, Y_t)}{h^T(0, y)}.\end{aligned}\tag{4.11}$$

The second equality follows from the fact that h^T solves the differential equation in (2.21), combined with Itô's formula. The first equality follows from the identity, for any adapted, integrable processes a, b and Wiener process W, that

$$\mathcal{E}\left(\int(a_s + b_s)\,dW_s\right) = \mathcal{E}\left(\int a_s\,dW_s\right)\mathcal{E}\left(\int b_s\,dW_s - \int b_s a_s\,ds\right).$$

Using (4.11) and (4.10) in (4.9) and applying Proposition 4.4 gives

$$\mathbb{E}^{\mathbb{P}^y}\big[D_T^{v^T}\big] = \mathbb{E}^{\hat{\mathbb{P}}^y}\left[\frac{h^T(T, Y_T)}{h^T(0, y)}\right] = 1,$$

which is the desired result. □

4.4 Densities and wealth processes

The last prerequisite for the main result is to relate the terminal wealths resulting from using the finite-horizon optimal strategies π^T of (2.15) and the long-run optimal strategy $\hat{\pi}$ of (2.18). A calculation similar to (4.11) using (2.16), (2.19) and (4.8) gives

$$\frac{D_t^{v^T}}{D_t^{\hat{v}}} = \frac{h^T(t, Y_t)}{h^T(0, y)}\mathcal{E}\left(-\int a\frac{h_y^T}{h^T}\Delta\, d\hat{B}\right)_t, \tag{4.12}$$

where

$$\Delta := q\delta\rho'\bar{\rho} \tag{4.13}$$

and the Brownian motion $\hat{B}$ comes from (4.2) and is related to B by the equality $d\hat{B}_t = dB_t + (q\overline{\rho}\sigma'\Sigma^{-1}\mu + \Delta' a\hat{v}_y/\hat{v})\,dt$.

Lemma A.3 in [22] implies

$$\begin{aligned}(X_t^{0,T})^p\big(v^T(t, Y_t)\big)^\delta &= \mathbb{E}^{\mathbb{P}^y}\big[(X_T^{0,T})^p \big| X_t^{0,T}, Y_t\big] = \mathbb{E}^{\mathbb{P}^y}\big[(X_T^{0,T})^p\big|\mathcal{F}_t\big],\\ &= \mathbb{E}^{\mathbb{P}^y}\big[(X_T^{0,T})^p\big]\mathbb{E}^{\mathbb{P}^y}\big[D_T^{v^T}\big|\mathcal{F}_t\big] = \big(v^T(0, y)\big)^\delta D_t^{v^T},\end{aligned} \tag{4.14}$$

where the last equality follows since D^{v^T} is a $\mathbb{P}^y$-martingale on $[0, T]$ and by the definition of $v^T(0, y)$. Note that $\hat{v}$ from [21, Eq. (75)] is equal to $\delta \log \hat{v}$ here and λ from [21, Eq. (75)] is equal to $\delta\lambda_c$ here. So we obtain

$$(\hat{X}_t)^p\big(\hat{v}(Y_t)\big)^\delta = e^{\delta\lambda_c t}\big(\hat{v}(y)\big)^\delta D_t^{\hat{v}}.$$

Therefore, since $v^T(t, y) = e^{\lambda_c(T-t)}\hat{v}(y)h^T(t, y)$,

$$\begin{aligned}\frac{X_t^{0,T}}{\hat{X}_t} &= \left(\frac{(v^T(0, y))^\delta D_t^{v^T}(\hat{v}(Y_t))^\delta}{(v^T(t, Y_t))^\delta e^{\delta\lambda_c t}(\hat{v}(y))^\delta D_t^{\hat{v}}}\right)^{1/p} = \left(\frac{D_t^{v^T}}{D_t^{\hat{v}}}\right)^{1/p}\left(\frac{h^T(t, Y_t)}{h^T(0, y)}\right)^{-\delta/p}\\ &= \left(\frac{h^T(t, Y_t)}{h^T(0, y)}\right)^{\frac{1-\delta}{p}}\mathcal{E}\left(-\int a\frac{h_y^T}{h^T}\Delta\, d\hat{B}_t^{\frac{1}{p}}\right),\end{aligned} \tag{4.15}$$

where the last equality uses (4.12). Equations (4.12) and (4.15) will be used in the next section.

Remark 4.7 The proof of Proposition 4.6 showed that D^{v^T} is a $(\mathbb{P}^y, (\mathcal{F}_t)_{0\le t\le T})$-martingale for each $y \in E$. Thus, (4.14) implies that

$$D_t^{v^T} = \left.\frac{d\mathbb{P}^{T,y}}{d\mathbb{P}^y}\right|_{\mathcal{F}_t}. \tag{4.16}$$

4.5 Proof of main results in Sects. 2.3.4 and 2.3.5

Proof of Proposition 2.15 By Theorem 18 in [21], under Assumptions 2.8 and 2.9, the decay condition in (2.28) yields the existence of a function $\hat{v}$ which satisfies (2.17), (2.24), along with the first inequality in (2.25). By Hölder's inequality, (2.27) ensures that the second inequality in (2.25) holds as well, proving the assertion. □

Proof of Lemma 2.17 Recall the notation of Sect. 4.4. From (4.10), (4.16) and (4.12), the limit in (2.30) holds provided that

$$\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty} \frac{h^T(t,Y_t)}{h^T(0,y)}\,\mathcal{E}\left(-\int a\frac{h_y^T}{h^T}\Delta\,d\hat{B}\right)_t = 1, \tag{4.17}$$

where Δ is from (4.13). Set $L_t^T = h^T(t,Y_t)/h^T(0,y)$. Proposition 4.4 implies that for each T, L^T is a positive $\hat{\mathbb{P}}^y$-martingale on $[0,T]$ with expectation 1, and for each $t\geq 0$, we have $\lim_{T\to\infty} L_t^T = 1$ $\hat{\mathbb{P}}^y$-almost surely. Therefore, Fatou's lemma gives $1 \geq \lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}[L_t^T] \geq \mathbb{E}^{\hat{\mathbb{P}}^y}[\liminf_{T\to\infty} L_t^T] = 1$, which implies by Scheffé's lemma that $\lim_{T\to\infty}\mathbb{E}^{\hat{\mathbb{P}}^y}[|L_t^T - 1|] = 0$. As shown in (4.11), we have $L_t^T = \mathcal{E}(\int a h_y^T/h^T\,d\hat{W})_t$. Lemma 3.9 thus yields

$$\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\left[\int a\frac{h_y^T}{h^T}\,d\hat{W}, \int a\frac{h_y^T}{h^T}\,d\hat{W}\right]_t = 0.$$

Observing that $\|\Delta\|^2$ is a constant, the previous identity implies that

$$\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\left[\int a\frac{h_y^T}{h^T}\Delta\,d\hat{B}, \int a\frac{h_y^T}{h^T}\Delta\,d\hat{B}\right]_t = 0,$$

whence $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\int_0^t a\frac{h_y^T}{h^T}\Delta\,d\hat{B} = 0$, which implies

$$\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty}\mathcal{E}\left(\int a\frac{h_y^T}{h^T}\Delta\,d\hat{B}\right)_t = 1,$$

i.e., the second factor on the left-hand side of (4.17) also converges to 1. This concludes the proof of (4.17). □

Proof of Theorem 2.18 Let $\varepsilon > 0$, and let $A_T \in \mathcal{F}_t$ denote either of the two events $\{\sup_{u\in[0,t]}|r_u^T - 1| \geq \varepsilon\}$, $\{[\Pi^T, \Pi^T] \geq \varepsilon\}$. According to Proposition 2.5, we have $\lim_{T\to\infty}\mathbb{P}^{T,y}[A_T] = 0$. Lemma 2.17 shows that $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty} d\mathbb{P}^{T,y}/d\hat{\mathbb{P}}^y|_{\mathcal{F}_t} = 1$ for $t \geq 0$ and $y \in E$. Lemma 4.2 shows that $\hat{\mathbb{P}}^y$ and $\mathbb{P}^y$ are equivalent on $\mathcal{F}_t$ for $t \geq 0$. Thus, the result follows by Lemma 2.6 taking $\mathbb{Q}^T = \mathbb{P}^{T,y}$, $\tilde{\mathbb{Q}} = \hat{\mathbb{P}}^y$ and $\mathbb{Q} = \mathbb{P}^y$. □

Proof of Theorem 2.19 A similar argument as in the proof of Lemma 2.17, combined with (4.15), yields that $\hat{\mathbb{P}}^y\text{-}\lim_{T\to\infty} X_t^{0,T}/\hat{X}_t = 1$. On the other hand, Theorem 2.18(a) implies that $\hat{\mathbb{P}}^y$-$\lim_{T\to\infty} X_t^{1,T}/X_t^{0,T} = 1$ by using the equivalence between $\mathbb{P}^y$ and $\hat{\mathbb{P}}^y$. Hence the last two identities combined show that $\hat{\mathbb{P}}^y$-$\lim_{T\to\infty} \hat{r}_t^T = 1$. Now recall that $\hat{\pi}$ is the optimal portfolio for the logarithmic investor under $\hat{\mathbb{P}}^y$. It then follows from the numéraire property of $\hat{X}$ that $\hat{r}^T$ is a $\hat{\mathbb{P}}^y$-supermartingale, which implies that $\lim_{T\to\infty} \mathbb{E}^{\hat{\mathbb{P}}^y}[|\hat{r}_t^T - 1|] = 0$ by Fatou's lemma and Scheffé's lemma. As a result, the statements follow by applying Lemma 3.9 under the probability $\hat{\mathbb{P}}^y$, and they remain valid under the equivalent probability $\mathbb{P}^y$ (as in Lemma 2.6). □

Proof of Corollary 2.20 Given any $t > 0$ and a compact domain $D \subset E$ with smooth boundary, recall that $\lim_{T\to\infty} h^T(t, y) = \int_E \hat{v}(z)\hat{m}(z)\,dz$ by (4.7). Moreover, h^T is bounded on $[0, t) \times D$ uniformly in T, since (4.6) holds (see also Remark 4.5) and $\hat{v}$ is continuous and strictly positive on D. Furthermore, h^T satisfies the differential equation $\partial_t h^T + \mathcal{L}^{\hat{v},0} h^T = 0$ which is uniformly elliptic in $[0, t) \times D$. It then follows from the Schauder interior estimate (see e.g. Theorem 15 in [18, Chap. 3]) that for any sequence of $(h^T)_{T>t}$, there exists a further subsequence, say $(h^{T_n})_{n\in\mathbb{N}}$, such that h^{T_n} (resp. $h_y^{T_n}$) converges to $\int_E \hat{v}(z)\hat{m}(z)\,dz$ (resp. 0), uniformly in any subdomain of $[0, t) \times D$. Taking derivatives with respect to y on both sides of $h^T(t, y) = v^T(t, y)/(e^{\lambda_c}(T - t)\hat{v}(y))$ yields

$$\frac{h_y^T}{h^T} = \frac{v_y^T}{v^T} - \frac{\hat{v}_y}{\hat{v}}.$$

It then follows that for any sequence of $(v^T)_{T>0}$, there exists a further subsequence $(v^{T_n})_{n\in\mathbb{N}}$ such that $v_y^{T_n}/v^{T_n}$ converges to $\hat{v}_y/\hat{v}$ locally uniformly in $[0, t) \times D$. However, this implies that the previous convergence must hold along the entire sequence of T. Otherwise, there exist $\epsilon, \delta > 0$, $\tilde{t} < t$, a subdomain $\tilde{D} \subset D$ and a subsequence $(T_m)_{m\in\mathbb{N}}$ such that $\max_{[\delta,\tilde{t}]\times\tilde{D}} |v_y^{T_m}/v^{T_m} - \hat{v}_y/\hat{v}| \geq \epsilon$ for each m. However, this contradicts the fact that there exists a further subsequence along which the previous norm converges to zero. As a result,

$$\lim_{T\to\infty} \frac{v_y^T}{v^T} = \frac{\hat{v}_y}{\hat{v}} \quad \text{locally uniformly in } [0, \infty) \times E,$$

since the choices of t and D are arbitrary. This confirms the statement after combination with (2.15) and (2.18). □

Proof of Lemma 2.24 Clearly, if $p < 0$ and $\mu_1 \neq 0$, then $\Lambda \geq 0$ and $\Theta > 0$. If $0 < p < 1$ and $\mu_0 = 0$, then $\Lambda = (b\theta - a^2/2)^2 \geq 0$, and $\Theta > 0$ is equivalent to $(b + qa\rho\mu_1)^2 + a^2 q\mu_1^2/\delta > 0$.

Given $\Lambda \geq 0$, $\Theta > 0$, the assertions that $(\hat{v}, \lambda_c)$ solve (2.17) and satisfy both (2.24) and the first equality in (2.25) all follow from [21, Proposition 27]. It thus remains to show that $\mathbf{A} > -1$ and $\mathbf{B} > 0$. Consider first the case when $p < 0$, $\mu_1 \neq 0$. Here, $q < 0$ implies $a^2 q\mu_0^2/\delta \geq 0$ which gives $\sqrt{\Lambda} \geq |b\theta - a^2/2 - qa\rho\mu_0|$ and hence $a^2\mathbf{A} = \sqrt{\Lambda} + (b\theta - a^2/2 - qa\rho\mu_0) \geq 0 > -a^2$. Similarly, since $a^2 q\mu_1^2/\delta > 0$ it follows that $\sqrt{\Theta} > |b + qa\rho\mu_1|$ and hence $a^2\mathbf{B} = \sqrt{\Theta} + (b + qa\rho\mu_1) > 0$. This completes part (i). For part (ii), assume that $0 < p < 1$ and $\mu_0 = 0, \mu_1 \neq 0$. Since $\Lambda = (b\theta - a^2/2)^2$, it clearly holds that $a^2\mathbf{A} = 2a^2(b\theta - a^2/2) \geq 0 > -a^2$. Lastly, note that $b + qa\rho\mu_1 = 0$ is incompatible with $(b + qa\rho\mu_1)^2 + a^2q\mu_1^2/\delta > 0$ since $q < 0$. Thus, assume $b + qa\rho\mu_1 \neq 0$ and set $R = a^2q\mu_1^2/(\delta(b + qa\rho\mu_1)^2)$. Note that $q < 0$ and $(b + qa\rho\mu_1)^2 + a^2q\mu_a^2/\delta > 0$ imply $-1 < R < 0$. Furthermore,

$$\mathbf{B} = |b + qa\rho\mu_1|\big(\sqrt{1 + R} + \operatorname{sign}(b + qa\rho\mu_1)\big),$$

and hence $\mathbf{B} > 0$ if and only if $b + qa\rho\mu_1 > 0$. This completes part (ii). □

Acknowledgements We are grateful to the Associate Editor and two anonymous referees for carefully reading this paper and providing valuable suggestions, which greatly assisted us in improving this paper.

Paolo Guasoni is partially supported by the ERC (278295), NSF (DMS-0807994, DMS-1109047), SFI (07/MI/008, 07/SK/M1189, 08/SRC/FMC1389), and FP7 (RG-248896). Constantinos Kardaras is partially supported by the NSF DMS-0908461. Hao Xing is partially supported by an LSE STICERD grant.

References

1. Benninga, S., Mayshar, J.: Heterogeneity and option pricing. Rev. Deriv. Res. **4**, 7–27 (2000)
2. Bichteler, K.: Stochastic Integration with Jumps. Encyclopedia of Mathematics and Its Applications, vol. 89. Cambridge University Press, Cambridge (2002)
3. Bielecki, T.R., Pliska, S.R.: Risk sensitive asset management with transaction costs. Finance Stoch. **4**, 1–33 (2000)
4. Bielecki, T.R., Hernandez-Hernandez, D., Pliska, S.R.: Risk sensitive asset management with constrained trading strategies. In: Yong, J. (ed.) Recent Developments in Mathematical Finance, Shanghai, 2001, pp. 127–138. World Scientific, River Edge (2002)
5. Cheridito, P., Summer, C.: Utility maximization under increasing risk aversion in one-period models. Finance Stoch. **10**, 147–158 (2006)
6. Cheridito, P., Filipović, D., Yor, M.: Equivalent and absolutely continuous measure changes for jump-diffusion processes. Ann. Appl. Probab. **15**, 1713–1732 (2005)
7. Cox, J.C., Huang, C.: A continuous-time portfolio turnpike theorem. J. Econ. Dyn. Control **16**, 491–507 (1992)
8. Cvitanić, J., Malamud, S.: Price impact and portfolio impact. J. Financ. Econ. **100**, 201–225 (2011)
9. Delbaen, F., Schachermayer, W.: A general version of the fundamental theorem of asset pricing. Math. Ann. **300**, 215–250 (1994)
10. Delbaen, F., Schachermayer, W.: The fundamental theorem of asset pricing for unbounded stochastic processes. Math. Ann. **312**, 463–520 (1998)
11. Detemple, J., Rindisbacher, M.: Dynamic asset allocation: portfolio decomposition formula and applications. Rev. Financ. Stud. **23**, 25–100 (2010)
12. Dybvig, P.H., Rogers, L.C.G., Back, K.: Portfolio turnpikes. Rev. Financ. Stud. **12**, 165–195 (1999)
13. Evans, L.C.: Partial Differential Equations. Graduate Studies in Mathematics, vol. 19. American Mathematical Society, Providence (1998)
14. Feller, W.: Two singular diffusion problems. Ann. Math. **54**, 173–182 (1951)
15. Fleming, W.H., McEneaney, W.M.: Risk-sensitive control on an infinite time horizon. SIAM J. Control Optim. **33**, 1881–1915 (1995)
16. Fleming, W.H., Sheu, S.J.: Risk-sensitive control and an optimal investment model. Math. Finance **10**, 197–213 (2000)
17. Fleming, W.H., Sheu, S.J.: Risk-sensitive control and an optimal investment model. II. Ann. Appl. Probab. **12**, 730–767 (2002)
18. Friedman, A.: Partial Differential Equations of Parabolic Type. Prentice-Hall, Englewood Cliffs (1964)
19. Friedman, A.: Stochastic Differential Equations and Applications. Vol. 1. Probability and Mathematical Statistics, vol. 28. Academic Press/Harcourt Brace Jovanovich, New York (1975)

20. Goll, T., Kallsen, J.: A complete explicit solution to the log-optimal portfolio problem. Ann. Appl. Probab. **13**, 774–799 (2003)
21. Guasoni, P., Robertson, S.: Portfolios and risk premia for the long run. Ann. Appl. Probab. **22**, 239–284 (2012)
22. Guasoni, P., Robertson, S.: Static fund separation of long-term investments (2013). Math. Financ. Available at http://onlinelibrary.wiley.com/doi/10.1111/mafi.12017/abstract
23. Hakansson, N.H.: Convergence to isoelastic utility and policy in multiperiod portfolio choice. J. Financ. Econ. **1**, 201–224 (1974)
24. Heath, D., Schweizer, M.: Martingales versus PDEs in finance: an equivalence result with examples. J. Appl. Probab. **37**, 947–957 (2000)
25. Huang, C.-F., Zariphopoulou, T.: Turnpike behavior of long-term investments. Finance Stoch. **3**, 15–34 (1999)
26. Huberman, G., Ross, S.: Portfolio turnpike theorems, risk aversion, and regularly varying utility functions. Econometrica **51**, 1345–1361 (1983)
27. Jacod, J., Shiryaev, A.N.: Limit Theorems for Stochastic Processes, 2nd edn. Grundlehren der Mathematischen Wissenschaften, vol. 288. Springer, Berlin (2003). [Fundamental Principles of Mathematical Sciences]
28. Jin, X.: Consumption and portfolio turnpike theorems in a continuous-time finance model. J. Econ. Dyn. Control **22**, 1001–1026 (1998)
29. Kabanov, Y., Kramkov, D.: Asymptotic arbitrage in large financial markets. Finance Stoch. **2**, 143–172 (1998)
30. Kaise, H., Sheu, S.J.: On the structure of solutions of ergodic type Bellman equation related to risk-sensitive control. Ann. Probab. **34**, 284–320 (2006)
31. Kallsen, J.: Optimal portfolios for exponential Lévy processes. Math. Methods Oper. Res. **51**, 357–374 (2000)
32. Karatzas, I., Kardaras, C.: The numéraire portfolio in semimartingale financial models. Finance Stoch. **11**, 447–493 (2007)
33. Karatzas, I., Žitković, G.: Optimal consumption from investment and random endowment in incomplete semimartingale markets. Ann. Probab. **31**, 1821–1858 (2003)
34. Kardaras, C.: The continuous behavior of the numéraire portfolio under small changes in information structure, probabilistic views and investment constraints. Stoch. Process. Appl. **120**, 331–347 (2010)
35. Kramkov, D., Sîrbu, M.: On the two-times differentiability of the value functions in the problem of optimal investment in incomplete markets. Ann. Appl. Probab. **16**, 1352–1384 (2006)
36. Kramkov, D., Sîrbu, M.: Sensitivity analysis of utility-based prices and risk-tolerance wealth processes. Ann. Appl. Probab. **16**, 2140–2194 (2006)
37. Kramkov, D., Sîrbu, M.: Asymptotic analysis of utility-based hedging strategies for small number of contingent claims. Stoch. Process. Appl. **117**, 1606–1620 (2007)
38. Larsen, K., Žitković, G.: Stability of utility-maximization in incomplete markets. Stoch. Process. Appl. **117**, 1642–1662 (2007)
39. Leland, H.: On turnpike portfolios. In: Szego, G., Shell, K. (eds.) Mathematical Methods in Investment and Finance, p. 24. North-Holland, Amsterdam (1972)
40. Mossin, J.: Optimal multiperiod portfolio policies. J. Bus. **41**, 215–229 (1968)
41. Nagai, H., Peng, S.: Risk-sensitive optimal investment problems with partial information on infinite time horizon. In: Yong, J. (ed.) Recent Developments in Mathematical Finance, Shanghai, 2001, pp. 85–98. World Scientific, River Edge (2002)
42. Nagai, H., Peng, S.: Risk-sensitive dynamic portfolio optimization with partial information on infinite time horizon. Ann. Appl. Probab. **12**, 173–195 (2002)
43. Najnudel, J., Nikeghbali, A.: A new kind of augmentation of filtrations. ESAIM Probab. Stat. **15**, 39–57 (2011)
44. Pinchover, Y.: Large time behavior of the heat kernel and the behavior of the Green function near criticality for nonsymmetric elliptic operators. J. Funct. Anal. **104**, 54–70 (1992)
45. Pinchover, Y.: Large time behavior of the heat kernel. J. Funct. Anal. **206**, 191–209 (2004)
46. Pinsky, R.G.: Positive Harmonic Functions and Diffusion. Cambridge Studies in Advanced Mathematics, vol. 45. Cambridge University Press, Cambridge (1995)
47. Rogers, L.C.G., Williams, D.: Diffusions, Markov Processes, and Martingales. Vol. 2. Itô Calculus, Cambridge Mathematical Library. Cambridge University Press, Cambridge (2000). Reprint of the second (1994) edition
48. Ross, S.: Portfolio turnpike theorems for constant policies. J. Financ. Econ. **1**, 171–198 (1974)
49. Zariphopoulou, T.: A solution approach to valuation with unhedgeable risks. Finance Stoch. **5**, 61–82 (2001)